An elementary approach to $x^2 + 7 = 2^n$

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Abstract

In this paper, we prove that the positive integer solutions of the equation $x^2 + 7 = 2^n$ are $x = 1, 3, 5, 11, 181$, corresponding to $n = 3, 4, 5, 7, 15$.

1 Introduction

The equation $x^2 + 7 = 2^n$ is known as Ramanujan-Nagell equation. Ramanujan[5] conjectured that only five solutions exist just when $n = 3, 4, 5, 7$, and 15. It was first solved by Nagell[4] that there are only five solutions, namely, $(n, x) = (3, 1), (4, 3), (5, 5), (7, 11), (15, 181)$. His method is using unique prime factorization in the field $\mathbb{Q}(\sqrt{-7})$. Cohn[2] solved $x^2 + D = y^n$ for 77 values of D in the range $1 \le D \le 100$ using unique prime factorization in the field $\mathbb{Q}(\sqrt{-D})$. Bugeaud[1], Mignotte, and Siksek solved $x^2 + D = y^n$ with $n \ge 3$ for $1 \le D \le 100$. Their method is the linear forms in logarithms and the modular approach.

We prove the problem by finding the integer points on elliptic curves that are related to the equation $x^2 + D = 2^n$.

Lemma 1. *The equation $x^2 + D = 2^n$ can be reduced to finding the integer points on elliptic curves. We take the three cases $n = 3a,\ n = 3a + 1,$ and $n = 3a + 2$.*

1) $n = 3a$
Let $y = 2^a$, then we get
$x^2 = y^3 - D$.

2) $n = 3a + 1$
$x^2 = 2y^3 - D$.
Equivalently, $X^2 = Y^3 - 4D$ *with* $(X, Y) = (2x, 2y)$.

3) $n = 3a + 2$
$x^2 = 4y^3 - D$.
Equivalently, $X^2 = Y^3 - 16D$ *with* $(X, Y) = (4x, 4y)$.

Theorem 1. *The positive integer solutions of the equation $x^2 + 7 = 2^n$ are $x = 1, 3, 5, 11, 181$, corresponding to $n = 3, 4, 5, 7, 15$.*

Proof.

$$x^2 + 7 = 2^n \tag{1}$$

From Lemma 1, we take the three cases $n = 3a,\ n = 3a + 1$, and $n = 3a + 2$.

1) $n = 3a$
Let $y = 2^a$, then we consider $x^2 = y^3 - 7$.
According to Magma[3], this elliptic curve has integral points $(y, x) = (2, \pm 1),\ (32, \pm 181)$.
Then, we get $(n, x) = (3, 1),\ (15, 181)$.

2) $n = 3a + 1$
Then, we consider $X^2 = Y^3 - 28$.
This elliptic curve has integral points $(Y, X) = (4, \pm 6),\ (8, \pm 22),\ (37, \pm 225)$.
We get $(n, x) = (4, 3),\ (7, 11)$.

3) $n = 3a + 2$
Similarly, $X^2 = Y^3 - 112$.
This elliptic curve has integral point $(Y, X) = (8, \pm 20)$.
We get $(n, x) = (5, 5)$.

Hence, there are only five integral solutions $(n, x) = (3, 1),\ (4, 3),\ (5, 5),\ (7, 11),\ (15, 181)$. □

References

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[5] S. Ramanujan, Question 464, J. Indian Math. Soc. 5 (1913), 120.