About the root-finding problem with applications

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Abstract

We propose necessary and sufficient conditions for the root-finding problem. A quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ is studied in term of the phase kickback as an application of the necessary and sufficient condition. As a result, we find a simple formula for the root-finding problem. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. We expect our discussions give some insight for future studies for root-finding problem.

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I. INTRODUCTION

The great success of quantum mechanics (cf. [1–7]) is recognized by the scientific community for physical theories. Between the articles of research for constructing theoretical quantum algorithms [8] it may be mentioned as follows: In 1985, the Deutsch algorithm was introduced and constructed for the function property problem [9–11]. In 1993, the Bernstein–Vazirani algorithm was proposed for identifying linear functions [12, 13]. Generalization of the Bernstein–Vazirani algorithm beyond qubit systems is reported [14]. In 1994, Simon's algorithm [15] and Shor's algorithm [16] were discussed for period finding of periodic functions. In 1996, Grover [17] provided an algorithm for unordered object finding and the motivation for exploring the computational possibilities offered by quantum mechanics. In 2020, a parallel computation for all of the combinations of values in variables of a logical function was proposed by Nagata and Nakamura [18]. In 2021, concrete quantum circuits for addition of two numbers of arbitrary length were discussed by Nakamura and Nagata [19].

Continuous-variable quantum information is the area of quantum information science that makes use of physical observables, such as the strength of an electromagnetic field, whose numerical values belong to continuous intervals. In 1998, Braunstein studied error correction for continuous quantum variables [20] and quantum error correction for communication with linear optics [21]. In 1999, Lloyd and Braunstein proposed quantum computation over continuous variables [22]. The same year, Ralph considered continuous-variable quantum cryptography [23]. In 2000, Hillery discussed quantum cryptography with squeezed states [24], while Reid described quantum cryptography with a predetermined key using continuous-variable Einstein-Podolsky-Rosen correlations [25].

In 2001, secure quantum key distribution using squeezed states was studied by Gottesman and Preskill [26]. A year later, continuous-variable quantum cryptography using coherent states was first proposed by Grosshans and Grangier [27]. Efficient classical simulation of continuous-variable quantum information processes is studied by Bartlett, Sanders, Braunstein, and Nemoto [28]. Continuous-variable quantum computing and its applications to cryptography are discussed by Diep, Nagata, and Wong [29].

Recently, Nagata and Nakamura discuss a quantum algorithm for finding the roots of a polynomial function by using the generalized Bernstein–Vazirani algorithm [30]. However, they restrict themselves to an assumption that all the roots are in the integers $\mathbf{Z}$. Here, all the roots considered here are in the real numbers $\mathbf{R}$. All the roots are different numbers. How do we find all the roots of the polynomial function? We expect our discussions give some insight for future studies for root-finding problem.

In this paper, we propose necessary and sufficient conditions for the root-finding problem. A quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ is studied in term of the phase kickback as an application of the necessary and sufficient condition. As a result, we find a simple formula for the root-finding problem. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. We expect our discussions give some insight for future studies for root-finding problem.

II. NECESSARY AND SUFFICIENT CONDITIONS FOR THE ROOT-FINDING PROBLEM

Let us consider necessary and sufficient conditions for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. That is, $|r_1| < |r_2| < ... < |r_m|, r_j \in \mathbf{R}$, $f(x) \in \mathbf{R}$, $x \in \mathbf{R}$, and $a_j \in \mathbf{R}$. $|r_j|$ is the absolute value of the root r_j of the function. Here the problem is of searching necessary and sufficient conditions for finding the roots of the polynomial function. We introduce a natural number d and suppose the following relation:

$$d \geq 2. \tag{1}$$

Let us discuss the structure of quantum computing. To this end, we introduce the transformation U_f (using the polynomial function f) defined by the mapping

$$U_f|x\rangle|j\rangle = |x\rangle|(|f(x)| + j) \bmod d\rangle, \tag{2}$$

where $|f(x)|$ is the absolute value of $f(x)$.

We define a quantum state $|\phi_d\rangle$ as follows:

$$|\phi_d\rangle = \frac{1}{\sqrt{d}} \int_0^d dj \omega(d)^{d-j}|j\rangle, \tag{3}$$

where $\omega(d) = e^{2\pi i/d}$. By the phase kickback [31] (See Appendix A) we have the following formula:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{4}$$

Notice that

$$(U_f)^d|x\rangle|j\rangle = |x\rangle|(d|f(x)| + j) \bmod d\rangle = |x\rangle|j\rangle. \tag{5}$$

Therefore, the mapping U_f is a cyclic transformation.

Here, we define the input state as follows:

$$|\psi\rangle_d = \frac{1}{\sqrt{d}} \int_{-\infty}^{+\infty} dx|x\rangle|\phi_d\rangle. \tag{6}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \frac{1}{\sqrt{d}} \int_{-\infty}^{+\infty} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{7}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$.

Again, we define the input state as follows (d and e are relatively prime and $d < e$):

$$|\psi\rangle_e = \frac{1}{\sqrt{e}} \int_{-\infty}^{+\infty} dx|x\rangle|\phi_e\rangle. \tag{8}$$

By applying U_f, to $|\psi_0\rangle$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_e = \frac{1}{\sqrt{e}} \int_{-\infty}^{+\infty} dx\omega(e)^{|f(x)|}|x\rangle|\phi_e\rangle. \tag{9}$$

Thus, by looking at the state $U_f|\psi\rangle_e$, we see the phase factor $\omega(e)^{|f(x)|}$.

We have several necessary and sufficient conditions for finding all the roots of a polynomial function.

Theorem

$$\begin{aligned} & |f(r)| = 0 \\ & \Leftrightarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \\ & \Leftrightarrow U_f = I \\ & \Leftrightarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e, \end{aligned} \tag{10}$$

where d and e are relatively prime and $d < e$, I is an identity operator, and r is a root of $f(x)$.

Proposition 1

$$|f(r)| = 0 \Rightarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1. \tag{11}$$

Proof: If $|f(r)| = 0$, then $\omega(d)^0 = 1$ and $\omega(e)^0 = 1$.

QED

Proposition 2

$$|f(r)| = 0 \Leftarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1. \tag{12}$$

Proof: If $\omega(d)^{|f(r)|} = 1$, then $|f(r)| = 0$ or $|f(r)| = dp, (p = 1, 2, 3, ...)$. If $\omega(e)^{|f(r)|} = 1$, then $|f(r)| = 0$ or $|f(r)| = eq, (q = 1, 2, 3, ...)$. d and e are relatively prime and $d < e$. Thus $|f(r)| = dp$ and $|f(r)| = eq$ are not realized. Therefore, $\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1$ implies $|f(r)| = 0$.

QED

Proposition 3

$$\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \Leftarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e. \tag{13}$$

Proof: We define the input state as follows:

$$|\psi\rangle_d = \frac{1}{\sqrt{d}} \int_{-\infty}^{+\infty} dx|x\rangle|\phi_d\rangle. \tag{14}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \frac{1}{\sqrt{d}} \int_{-\infty}^{+\infty} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{15}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$. Thus, we have

$$U_f|\psi\rangle_d = |\psi\rangle_d \Rightarrow \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle = \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx|x\rangle|\phi_d\rangle \Rightarrow \omega(d)^{|f(x)|} = 1. \tag{16}$$

Similarly, we have, using e,

$$U_f|\psi\rangle_e = |\psi\rangle_e \Rightarrow \omega(e)^{|f(x)|} = 1. \tag{17}$$

Therefore, $\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \Leftarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e$.
QED
Proposition 4

$$U_f = I \Rightarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e. \tag{18}$$

Proof: If $U_f = I$, then $U_f|\psi\rangle_d = |\psi\rangle_d$ and $U_f|\psi\rangle_e = |\psi\rangle_e$.
QED
Proposition 5

$$|f(r)| = 0 \Rightarrow U_f = I. \tag{19}$$

Proof: If $|f(r)| = 0$, then $U_f|r\rangle|j\rangle = |r\rangle|(|f(r)| + j) \bmod d\rangle = |r\rangle|j\rangle$.
QED
Thus, we prove the theorem (10). We expect our discussions give some insight for future studies for root-finding problem.

III. APPLICATION OF THE NECESSARY AND SUFFICIENT CONDITION

Let us consider a quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$. We use necessary and sufficient conditions for finding all the m roots of the polynomial function. See the theorem (10). Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. That is, $|r_1| < |r_2| < ... < |r_m|$. $|r_j|$ is the absolute value of the root r_j of the function. Here the problem is of searching quantum algorithm for finding the roots of the polynomial function.

We define a quantum state $|\phi_d\rangle$ as follows:

$$|\phi_d\rangle = \frac{1}{\sqrt{d}}\int_0^d dj\omega(d)^{d-j}|j\rangle, \tag{20}$$

where $\omega(d) = e^{2\pi i/d}$. By the phase kickback [31] (See Appendix A) we have the following formula:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{21}$$

Here, we define the input state as follows:

$$|\psi\rangle_d = \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx|x\rangle|\phi_d\rangle. \tag{22}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{23}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$. If r is a root of $f(x)$, then $f(r) = 0$. Thus, by looking at the state $U_f|\psi\rangle_d$, we do not see the phase factor $\omega(d)^{|f(r)|}$. There are m points, in the interval $[-\infty, +\infty]$, such that

$$U_f|\psi\rangle_d = |\psi\rangle_d. \tag{24}$$

Similarly, we have, using e,

$$U_f|\psi\rangle_e = |\psi\rangle_e. \tag{25}$$

Hence we have

$$(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) = \begin{cases} \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx\{\omega(d)^{|f(x)|} - 1\}|x\rangle|\phi_d\rangle \\ +\frac{1}{\sqrt{e}}\int_{-\infty}^{+\infty} dx\{\omega(e)^{|f(x)|} - 1\}|x\rangle|\phi_e\rangle & \text{if} \quad f(x) \neq 0, \\ \mathbf{0} & \text{if} \quad f(x) = 0. \end{cases} \tag{26}$$

Therefore, we determine all the m roots by evaluating the relation (26). As a result, we find a simple formula for the root-finding problem (26). Let r be a root. Then $f(r) = 0$, thus the oracle becomes the identity operator I. That is, $U_f = U_0 = I$ if f is zero. In more detail, the phase kickback occurs if f is not zero. The phase kickback does not occur if f is zero.

When $f(x) = 0$ we have

$$(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) = \mathbf{0}, \tag{27}$$

then, x is a root, that is, $x = r_k(k = 1, 2, ..., m)$. When $f(x) \neq 0$ we have

$$\begin{aligned}(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) &= \frac{1}{\sqrt{d}}\int_{-\infty}^{+\infty} dx\{\omega(d)^{|f(x)|} - 1\}|x\rangle|\phi_d\rangle \\ &+\frac{1}{\sqrt{e}}\int_{-\infty}^{+\infty} dx\{\omega(e)^{|f(x)|} - 1\}|x\rangle|\phi_e\rangle \quad (\neq \mathbf{0}),\end{aligned} \tag{28}$$

then, x is not a root. It seems that we need to determine the quantum state $((U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e))$, which we think a final procedure like quantum state tomography is needed. Then we can pick up the m roots. We expect our discussions give some insight for future studies for root-finding problem.

IV. CONCLUSIONS

We have proposed necessary and sufficient conditions for the root-finding problem. A quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ has been studied in term of the phase kickback as an application of the necessary and sufficient condition. As a result, we have found a simple formula for the root-finding problem. Here all the roots have been in the real numbers $\mathbf{R}$. All the roots have been different numbers and the number of the roots is m. We have expected our discussions give some insight for future studies for root-finding problem.

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Ethical approval

The authors are in an applicable thought to ethical approval.

Competing interests

The authors state that there is no conflict of interest.

Author contributions

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Data availability

No data associated in the manuscript.

Appendix A: The phase kickback

We have the following formula by the phase kickback [31]:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{A1}$$

where $\omega(d) = e^{2\pi i/d}$ and $|f(x)|$ is the absolute value of $f(x)$.

In what follows, we discuss the rationale behind the above relation (A1). Consider the action of the U_f gate on the state $|x\rangle|\phi_d\rangle$. Each term in $|\phi_d\rangle$ is of the form $\omega^{d-j}|j\rangle$. We observe that

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{d-j}|x\rangle|(|f(x)| + j) \bmod d\rangle. \tag{A2}$$

A variable k is introduced such that $|f(x)| + j = k$, from which it follows that $d - j = d + |f(x)| - k$. Thus, (A2) becomes

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{|f(x)|}\omega^{d-k}|x\rangle|k \bmod d\rangle. \tag{A3}$$

If $k < d$ we have that $|k \bmod d\rangle = |k\rangle$ and thus the terms in $|\phi_d\rangle$ for which $k < d$ are transformed as follows:

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{|f(x)|}\omega^{d-k}|x\rangle|k\rangle. \tag{A4}$$

On the other hand, as both $|f(x)|$ and j are bounded from above by d, k is strictly less than $2d$. Thus, when $d \leq k < 2d$, we have $|k \bmod d\rangle = |k - d\rangle$. Let $k - d = m$. We have

$$\begin{aligned}\omega^{|f(x)|}\omega^{d-k}|x\rangle|k \bmod d\rangle &= \omega^{|f(x)|}\omega^{-m}|x\rangle|m\rangle\\ &= \omega^{|f(x)|}\omega^{d-m}|x\rangle|m\rangle.\end{aligned} \tag{A5}$$

Hence, the terms in $|\phi_d\rangle$ for which $k \geq d$ are transformed as follows:

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{|f(x)|}\omega^{d-m}|x\rangle|m\rangle. \tag{A6}$$

Finally, regarding (A4) and (A6), we have

$$U_f|x\rangle|\phi_d\rangle = \omega^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{A7}$$

Therefore, the relation (A1) holds.

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