A glimpse into Quantum Gravity : A Noncommutative Spacetime results in Quantization of Area, Mass and Entropy

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Abstract

It is shown how a Noncommutative spacetime leads to an area, mass and entropy quantization condition which allows to derive the Schwarzschild black hole entropy $\frac{A}{4G}$, the logarithmic corrections, and further corrections, from the discrete mass transitions taken place among different mass states in $D = 4$. The higher dimensional generalization of the results in $D = 4$ follow. The discretization of the entropy-mass relation $S = S(M)$ leads to an entropy quantization of the form $S = S(M_n) = n$, and such that one may always assign n "bits" to the discrete entropy, and in doing so, make contact with quantum information. The physical applications of mass quantization, like the counting of states contributing to the black hole entropy, black hole evaporation, and the direct connection to the black holes-string correspondence [23] via the asymptotic behavior of the number of partitions of integers, follows. To conclude, it is shown how the recent large N Matrix model (fuzzy sphere) of [20] leads to very similar results for the black hole entropy as the physical model described in this work and which based on the discrete mass transitions originating from the noncommutativity of the spacetime coordinates.

Keywords : Noncommutative Geometry; Gravity, Black Hole, Entropy; Strings; Matrix Models; Partitions.

1 Introduction : Noncommutative Spacetime and Black Hole Entropy from a Point Mass Source

The idea of a Quantum Spacetime where the spacetime coordinates do not commute was proposed early on by Heisenberg and Ivanenko as a way to eliminate infinities from Quantum Field Theory. Snyder published the first concrete example [1] of a noncommutative algebra involving the spacetime coordinates, and it was generalized shortly after by Yang [2], to include noncommuting momentum variables as well. We learnt from General Relativity that the Poincare algebra cannot be implemented on a curved spacetime, but only on its flat tangent space (Minkowski spacetime). The momentum operators don't commute on a curved spacetime. And vice versa, by Born's principle of reciprocity [3], [4] the coordinate operators do not commute on a curved *momentum* space. This prompted the formulation of Quantum Mechanics and Quantum Field Theory in Noncommutative spacetimes (also called Noncommutative QFT), and which might cast some light in the formulation of Quantum Gravity by encoding both key aspects of a curved and a noncommuting spacetime (a curved noncommuting spacetime).

Given a flat $6D$ spacetime with coordinates $Y^A = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$, and a metric $\eta_{AB} = diag(-1, +1, +1, \ldots, +1)$, the Yang algebra [2] can be derived in terms of the $so(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators

$$J^{AB} = -(Y^A\ \Pi^B\ -\ Y^B\ \Pi^A)\ =\ i\ Y^A \frac{\partial}{\partial Y_B}\ - i\ Y^B \frac{\partial}{\partial Y_A} \tag{1.1}$$

where $\Pi^A = -i(\partial/\partial Y_A)$ is the canonical conjugate momentum variable to Y^A. Their commutators are

$$[Y^A, Y^B]\ =\ 0,\ [\Pi^A, \Pi^B]\ =\ 0,\ \ [Y^A, \Pi^B]\ =\ i\,\eta^{AB},\ \ A, B = 1,2,3,4,5,6 \tag{1.2}$$

The coordinates Y^A commute. The momenta Π^A also commute, and Y^A, Π^B obey the Weyl-Heisenberg algebra in $6D$.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting $4D$ spacetime coordinates X^μ, the noncommuting momenta P^μ, and the Lorentz $so(5,1)$ algebra generators leading to the Yang algebra [2] is given by

$$\begin{aligned} X^\mu\ &\leftrightarrow\ L_P\ J^{\mu 5}\ =\ -\ L_P\ (Y^\mu\ \Pi^5\ -\ Y^5\ \Pi^\mu) \\ P^\mu\ &\leftrightarrow\ \frac{1}{\mathcal{L}}\ J^{\mu 6}\ =\ -\ \frac{1}{\mathcal{L}}\ (Y^\mu\ \Pi^6\ -\ Y^6\ \Pi^\mu),\ \ \mu, \nu = 1,2,3,4 \end{aligned} \tag{1.3}$$

and which requires the introduction of an ultra-violet cutoff scale L_P given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$ the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [1] and Yang [2] algebras.

The other generators are given by

One must include also the remaining $so(5,1)$ generators

$$\mathcal{N}\ \equiv\ J^{56} = -(Y^5\ \Pi^6 - Y^6\ \Pi^5),\ \ J^{\mu\nu}\ =\ -(Y^\mu\ \Pi^\nu - Y^\nu\ \Pi^\mu),\ \ \mu, \nu = 1,2,3,4 \tag{1.4}$$

One can then verify that the Yang algebra is recovered after imposing the above correspondence (1.3)

$$[X^\mu,\ X^\nu] \ = \ -i\ L_P^2\ J^{\mu\nu},\ \ [P^\mu,\ P^\nu] \ = \ -i\ (\frac{1}{\mathcal{L}})^2\ J^{\mu\nu},\ \ \eta^{55} \ = \ \eta^{66} = 1 \quad (1.5)$$

$$[X^\mu,\ J^{\nu\rho}] \ = \ i\ (\eta^{\mu\rho}\ X^\nu \ - \ \eta^{\mu\nu}\ X^\rho) \quad (1.6)$$

$$[P^\mu,\ J^{\nu\rho}] \ = \ i\ (\eta^{\mu\rho}\ P^\nu \ - \ \eta^{\mu\nu}\ P^\rho\ ) \quad (1.7)$$

$$[X^\mu,\ P^\nu] \ = \ -i\ \eta^{\mu\nu}\ \frac{L_P}{\mathcal{L}}\ \mathcal{N},\ \ [J^{\mu\nu}, \mathcal{N}] = 0 \quad (1.8)$$

$$[X^\mu,\ \mathcal{N}] \ = \ i\ L_P \mathcal{L}\ P^\mu,\ \ [P^\mu,\ \mathcal{N}] \ = \ -i\ \frac{1}{L_P \mathcal{L}}\ X^\mu \quad (1.9)$$

and where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $so(3,1)$ Lorentz algebra in $4D$. They are of the form

$$[\ J^{\mu_1\mu_2},\ J^{\nu_1\nu_2}\ ] \ = \ -i\ \eta^{\mu_1\nu_1}\ J^{\mu_2\nu_2}\ + i\ \eta^{\mu_1\nu_2}\ J^{\mu_2\nu_1}\ + \\ i\ \eta^{\mu_2\nu_1}\ J^{\mu_1\nu_2}\ - i\ \eta^{\mu_2\nu_2}\ J^{\mu_1\nu_1},\ \ \hbar = c = 1 \quad (1.10)$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of eq-(2.10) since the commutator of two Hermitian operators is anti-Hermitian. The $4D$ spacetime metric is $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

In [6] we discussed two approaches in the evaluation of the areal spectrum in $3D$ and associated with noncommutative coordinates that we labeled as operators as $\mathbf{x_i}; i = 1, 2, 3$. One approach was to write the operator $L_P^{-2} \sum_{i=1}^{i=3} \mathbf{x_i x^i}$ (in Planck units) as the difference $\sum_{i,j=1}^{i,j=4} \mathbf{J}_{ij}^2 - \sum_{i,j=1}^{i,j=3} \mathbf{J}_{ij}^2$ of the total orbital angular momentum squared in $D = 4$ and $D = 3$. So the eigenvalues can be obtained from the difference between the quadratic Casimirs of $SO(4)$ and $SO(3)$ given by $C_2[SO(4)] - C_2[SO(3)] = l_3(l_3 + 2) - l_2(l_2 + 1)$, where l_3 is the orbital angular momentum quantum number of the three-sphere S^3, and l_2 is the orbital angular momentum quantum number of the two-sphere S^2. In the very special case when $l_3 = l_2$ the difference $C_2[SO(4)] - C_2[SO(3)]$ is given by l_2 and such that $\sum_{i=1}^{i=3} \mathbf{x_i x^i} = l_2 L_P^2$ turns out to be *linear* in the angular momentum quantum number of the two-sphere $l_2 = l$.

However there is a subtlety because the eigenfunctions of the angular momentum operators associated with S^2 and S^3 are not the same. The eigenfunctions of the angular momentum operators $\mathbf{J}_{S^2}^2$ associated with S^2 are the spherical harmonics $Y_{lm}(\theta, \varphi)$ and which can be rewritten as $Y_{l_2 l_1}(\theta_2, \theta_1)$

$$Y_{l_2 l_1}(\theta_2, \theta_1) \ \equiv \ Y_{lm}(\theta, \varphi) \ = \ (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\ P_l^m(cos\theta)\ e^{im\varphi} \quad (1.11)$$

with $l_1 = m, l_2 = l; \theta_2 = \theta, \theta_1 = \varphi$ and where $P_{lm}(cos\theta)$ are the associated Legendre ploynomials.

The eigenfunctions of the angular momentum operators $\mathbf{J}^2_{S^3}$ associated with S^3 are given in terms of three angles $\theta_1 = \varphi, \theta_2 = \theta, \theta_3 = \xi$ and three quantum numbers l_1, l_2, l_3, obeying $l_3 \geq l_2 \geq |l_1|$, as follows [8]

$$Y_{l_1 l_2 l_3}(\theta, \varphi, \xi) \ = \ Y_{l_1 l_2}(\theta, \varphi) \ \sqrt{\frac{2l_3+2}{2} \ \frac{(l_3+l_2+1)!}{(l_3-l_2)!}} \ \sqrt{sin\xi} \ P^{-(l_2+\frac{1}{2})}_{l_3+\frac{1}{2}}(cos\xi) \tag{1.12}$$

where $P^{-(l_2+\frac{1}{2})}_{l_3+\frac{1}{2}}(cos\xi)$ is the associate Legendre function of the first kind that can be written in terms of the hypergeometric function ${}_2F_1$ as

$$P^{-(l_2+\frac{1}{2})}_{l_3+\frac{1}{2}}(cos\xi) \ \equiv \ \frac{1}{\Gamma(1+l_2+\frac{1}{2})} \ (\frac{1-cos\xi}{1+cos\xi})^{\frac{1}{2}(l_2+\frac{1}{2})} \ \times$$

$${}_2F_1\left(-(l_3+\frac{1}{2}), \ (l_3+\frac{1}{2})+1; \ 1+(l_2+\frac{1}{2}); \ \frac{1-cos\xi}{2}\right) \tag{1.13}$$

Note that because $Y_{l_1 l_2 l_3}(\theta, \varphi, \xi)$ factorizes $Y_{l_1 l_2}(\theta, \varphi) F_{l_3 l_2}(\xi)$, it can be seen also as an eigenfunction of $\mathbf{J}^2_{S^2}$ (the angular momentum operator associated with S^2) because $\mathbf{J}^2_{S^2} Y_{l_1 l_2 l_3}(\theta, \varphi, \xi) = l_2(l_2+1) Y_{l_1 l_2 l_3}(\theta, \varphi, \xi)$ due to the factorization property and the trivial fact that $\mathbf{J}^2_{S^2}$ does not act on the extra angle ξ.

Therefore one has

$$(\sum_{i=1}^{i=3} \ \mathbf{x_i x^i}) \ Y_{l_1 l_2 l_3} \ = \ L_P^2 \ (\mathbf{J}^2_{S^3} - \mathbf{J}^2_{S^2}) \ Y_{l_1 l_2 l_3} \ = \ L_P^2 \ [l_3(l_3+2) - l_2(l_2+1)] Y_{l_1 l_2 l_3} \tag{1.14}$$

giving $L_P^2 l_2 Y_{l_1 l_2 l_3}$ for the right hand side in the special case when $l_3 = l_2$. Since $4\pi r^2$ is the area of a sphere, when the coordinates are noncommutative, we can label $\mathbf{r}^2$ as the square of the radial operator, and the area spectrum of the quantum sphere is $4\pi L_P^2[l_3(l_3+2) - l_2(l_2+1)]$. The areal spectrum becomes *linear* in the angular momentum when $l_3 = l_2 = l$, so the areas are quantized in multiples of the Planck area, not unlike the Schwarzschild black hole horizon areas quantized in bits of Planck areas [5], This whole procedure can be repeated for the momentum, and in [6] we obtained the spectrum of the deformed quantum oscillator. The areal momentum is quantized in bits of a minimal areal momentum. Likewise, the areas were quantized in bits of a minimal Planck areas.

Recently it was explicitly shown [10] how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution. In order to achieve this, one requires to *extend* the domain of r to *negative* values $-\infty \leq r \leq +\infty$. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \geq 4$ after evaluating the non-vanishing Euclidean Einstein-Hilbert action. As usual, it was required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time.

In $D = 4$ the scalar curvature and the Euclidean action I turned out to be [10]

$$\mathcal{R} \;=\; \frac{4GM\delta(r)}{r^2} \;\Rightarrow\; I \;=\; -\;\frac{i}{16\pi G}\int_0^{\beta_H} d\tau \int_0^\infty \mathcal{R}\; 4\pi r^2\; dr \qquad (1.15)$$

the magnitude of the integral (1.15) becomes after inserting the inverse Hawking temperature $\beta = 8\pi GM$

$$|I| = \frac{1}{2}\; M\;\beta_H \;=\; 4\pi GM^2 \;=\; \frac{4\pi(2GM)^2}{4G} \;=\; \frac{4\pi r_h^2}{4G} \;=\; \frac{Area}{4L_P^2} \qquad (1.16)$$

and which is the Schwarzschild black hole entropy in $D > 4$.

In higher dimensions $D > 4$, the scalar curvature is [10]

$$\mathcal{R} \;=\; 2\;\frac{16\pi GM}{(D-2)\Omega_{D-2}}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \;=\; 2\; r_h^{D-3}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \qquad (1.17)$$

where $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$ is the solid angle of the $D-2$-dim hypersphere. The horizon radius is given by

$$r_h \;=\; \left(\frac{16\pi GM}{(D-2)\;\Omega_{D-2}}\right)^{\frac{1}{D-3}} \qquad (1.18)$$

and the magnitude of the Euclidean integral I

$$I \;=\; -\;\frac{i}{16\pi G}\int_0^{\beta} d\tau \int_0^\infty \mathcal{R}\;\Omega_{D-2}\; r^{D-2}\; dr \qquad (1.19)$$

after inserting the inverse Hawking temperature $\beta = 4\pi r_h/(D-3)$, becomes

$$|I| \;=\; \frac{\Omega_{D-2}\; r_h^{D-2}}{\mathbf{4}G_D} \;=\; \frac{\Omega_{D-2}}{\mathbf{4}G_D}\;\left(\frac{16\pi G_D M}{(D-2)\;\Omega_{D-2}}\right)^{\frac{D-2}{D-3}} \qquad (1.20)$$

which is the Schwarzschild black hole entropy in $D > 4$ dimensions given by one-quarter of the horizon area in Planck units. Essential in these findings was the result that $\int_0^\infty \delta(r)dr = \frac{1}{2}\int_{-\infty}^\infty \delta(r)dr = \frac{1}{2}$ resulting from the symmetry of the delta function $\delta(-r) = \delta(r)$.

The source of the black hole entropy is the point-mass. As the black hole evaporates completely its temperature *blows* up and this might be interpreted as a manifestation, reflection, of the initial spacetime singularity at $r = 0$ due to the presence of the point-mass source generating a scalar curvature $\mathcal{R} = 4GM\delta(r)/r^2 = \infty$ at $r = 0$, and $\mathcal{R} = 0$ for $r > 0$. Based on the key role that the point-mass plays in the derivation of the black hole entropy we shall combine it with the *noncommutativity* of the spacetime coordinates and show how this leads to the quantization of area, mass and entropy.

The outline of this work goes as follows. In the section **2** we shall recast the area-quantization condition, resulting from the *noncommutative* spacetime coordinates discussed in the introduction, in terms of a mass *quantization* condition and derive the black hole entropy $\frac{A}{4G}$, the logarithmic corrections, and further corrections, from the discrete mass transitions among the different mass states in $D = 4$. The higher dimensional generalization of the results in $D = 4$ follow. To conclude section **2**, we show that given an entropy-mass relation $S = S(M)$, the mass quantization conditions, and their discrete mass transitions, lead to an entropy quantization $S = S(M_n) = n$, such that one may assign n "bits" to this discrete entropy, and in doing so, make contact with quantum information.

Section **3** is devoted to the physical applications of the mass quantization, like the counting of states contributing to the black hole entropy, black hole evaporation, and the direct connection to the black holes-string correspondence [23] via the asymptotic behavior of the number of partitions of integers [29].

We conclude in section **4** by showing how that the large N Matrix model (fuzzy sphere [19]) approach of [20] leads to very similar results for the black hole entropy as the physical model described in this work and which is based on the discrete mass transitions originating from the *noncommutativity* of the spacetime coordinates and that resulted in the key quantization of area, mass and entropy. We shall employ throughout this work the units of $\hbar = c = k_B = 1$.

2 Black Hole Entropy from Discrete Mass Transitions

In this section we shall explore the physical implications behind the eigenvalues and eigenfunctions of the area-operators described in terms of the angular momentum operators of (hyper) spheres $\mathbf{S}^3, \mathbf{S}^2$ of the previous section. The main starting point is eq-(1.14). Let us begin with :

Case **A** : If one sets $l_3 = l_2 = n$ in eq-(1.14) it yields $\frac{\mathbf{r}^2}{L_P^2} Y_{l_1 l_2 l_3} = n Y_{l_1 l_2 l_3}$, with $Y_{l_1 l_2 l_3}$ given by eqs-(1.11,1.12,1.13). Given $G = L_P^2$ in $D = 4$, the area quantization of the spherical horizon of radius $r_h = 2GM$ can be recast also as a mass quantization condition as follows

$$r_h^2 \ = \ (2GM_n)^2 \ = \ n \ L_P^2 \ \Rightarrow \ \frac{4M_n^2}{m_P^2} \ = \ n \ \Rightarrow \ \frac{2M_n}{m_P} \ = \ \sqrt{n}, \quad m_P = (L_P)^{-1} \tag{2.1}$$

with n an integer $0, 1, 2, \ldots$. If there is a transition between two neighboring discrete mass states : $M_n \rightarrow M_{n-1}$, a thermal photon of energy $\omega_{n,n-1} = M_n - M_{n-1} = \Delta M_n$ is emitted (radiated), so that when $\Delta n = 1$ one learns from eq-(2.1), to a first order approximation, that

$$\frac{2\omega_{n,n-1}}{m_P} \;=\; \frac{2\Delta M_n}{m_P} \;\sim\; \frac{\Delta n}{2\sqrt{n}} \;=\; \frac{m_P}{4M_n},\ (\Delta n = 1) \tag{2.2}$$

leading to

$$\omega_{n,n-1} \;=\; \Delta M_n \;\sim\; \frac{m_P^2}{8M_n} \;=\; \frac{1}{8GM_n} \tag{2.3}$$

It is important to emphasize that if the transition occurs between states that are not neighbors, $\Delta n \neq 1$, one may inclined to claim erroneously that the frequencies of the photons emitted appear to be integer multiples of $\omega_{n,n-1}$. This is an artifact of the first order approximation in eq-(2.2). A more rigorous result reveals that the frequencies are *not* integer-multiples of the frequency $\omega_{n,n-1}$ of eq-(2.2), because the mass states $M_n \sim m_P\sqrt{n}$ are *not* equally spaced, like it occurs in the energy levels of a harmonic oscillator.

In the black body radiation spectrum, Wien's displacement law sates that the wavelength at which the intensity per unit wavelength of the radiation has a local maximum or peak, is only a function of the temperature and given by $\lambda_{peak} = \frac{b}{T}$, where the constant $b \simeq 2.897 \times 10^{-3}$ m-K is Wien's displacement constant [29]. Since frequency is inversely proportional to the wavelength, the peak frequency turns out to be directly proportional to the black body temperature.

Hence, if one *postulates* that the frequency is the same as the temperature, $\omega_{n,n-1} = T_n$, one finds that $T_n \sim \frac{1}{8GM_n}$ is inversely proportional to the mass M_n. The latter expression corresponds to a temperature whose functional form is $T(M) = \frac{1}{8GM}$ and agrees with the Hawking temperature $T_H = \frac{1}{8\pi GM}$ up to a factor of π. The entropy corresponding to a temperature $T = T(M) = \frac{1}{8GM}$ is defined as

$$S \;=\; \int \frac{dM}{T(M)} \;=\; \int dM\ (8GM) \;=\; 4GM^2 \;=\; \frac{4M^2}{m_P^2} \tag{2.4}$$

and this quadratic behavior in the mass matches the entropy of a black hole $\frac{4\pi(2GM)^2}{4G} = \frac{Area}{4G}$, up to a factor of π. One may note that a simple rescaling $L_P \rightarrow \frac{L_P}{\sqrt{\pi}}$ in the first term of eq-(2.1) suffices to obtain the *exact* expression for the Black Hole entropy. In other words, one has $\frac{A_n}{4G} = n\pi$ to be more precise. Given the Bekenstein-Hawking black hole entropy $S = \frac{4\pi M^2}{m_P^2}$, its *discretized* form becomes $S_n = \frac{4\pi M_n^2}{m_P^2} = n\pi$, and such that it is quantized in n-bits, in the same way that one-quarter of the horizon's area is quantized in integer multiples of Planck-area cells (up to a multiple of π). In the remaining of this work we shall *exclude* the factors of π for simplicity, keeping in mind that they should be taken into account.

Logarithmic corrections to the black hole entropy are obtained when one does *not* approximate the expression ΔM_n as displayed in eq-(2.2) but instead one evaluates *exactly* the mass increment ΔM_n by performing the binomial expansion in powers of $\frac{1}{n}$, with $n = \frac{4M_n^2}{m_P^2}$, as follows

$$\Delta M_n \ = \ \frac{m_P}{2} \ (\sqrt{n} \ - \ \sqrt{n-1}) \ = \ \frac{m_P}{2} \ \sqrt{n} \ \left(1 \ - \ \sqrt{1 - \frac{1}{n}} \right) \ \sim$$

$$\frac{m_P}{2} \ \sqrt{n} \ \left(\frac{1}{2n} \ + \ \frac{1}{8n^2} \ + \ \ldots \right) \tag{2.5}$$

Upon substituting $n = \frac{4M_n^2}{m_P^2}$ in eq-(2.5), which stems from the area/mass quantization, gives then for the *two* leading terms in the binomial expansion the following

$$\Delta M_n = \omega_{n,n-1} \ = \ T_n \ = \ \frac{1}{8GM_n} \ + \ \frac{1}{128G^2M_n^3} \tag{2.6}$$

and such discrete expression (2.6) corresponds to a temperature-mass relation of the form

$$T = T(M) \ = \ \frac{1}{8GM} \ + \ \frac{1}{128G^2M^3} \tag{2.7}$$

and one then obtains in this manner the first order corrections to the Hawking temperature (up to π factors). Hence, the logarithmic corrections to the black hole entropy are obtained from the integral

$$S \ = \ \int \frac{dM}{T(M)} \ = \ \int dM \ \left(\frac{1}{8GM} \ + \ \frac{1}{128G^2M^3} \right)^{-1} \ =$$

$$\frac{A}{4G} \ - \ \frac{1}{4} \ ln(\frac{A}{G} + 1) \tag{2.8a}$$

after inserting the expression for the horizon area $A = 4\pi(2GM)^2$ in terms of the mass M and inserting the factors of π judiciously. The discrete version of eq-(2.8a) is

$$S_n \ = \ \frac{A_n}{4G} \ - \ \frac{1}{4} \ ln(\frac{A_n}{G} + 1), \ \ \frac{A_n}{4G} \ = \ n\pi \ = \ \frac{4\pi M_n^2}{m_P^2} \tag{2.8b}$$

Higher order corrections to the Hawking temperature and black hole entropy follow by including the higher order terms in the binomial expansion.

A similar procedure to obtain the logarithmic corrections to the black hole entropy, after relating the frequency of the radiated photon to the temperature in discrete mass transitions, can be found in [13] and references therein. The mass spectrum of black holes has a long history, see [14], [16], [17], [18] among others. More recently, the quantum deformation of the Wheeler–DeWitt equation of a Schwarzchild black hole was studied by [13]. The quantum deformed black hole was based on a quantized model constructed from the quantum Heisenberg–Weyl $U_q(h_4)$ group. It was found that the event horizon area and the mass were quantized, degenerate, and bounded due to the nature of the quantum group when the deformation parameter was a root of unity.

There is an important and subtle remark that is in order. The reader may object to use the spectrum of the angular momentum operators to obtain the Schwarzschild black hole entropy when the black hole is *not* rotating. A Kerr rotating black hole is described in term of its mass M and a non-vanishing $J \neq 0$. The radius of the inner and outer horizons of a Kerr black hole are given by ($c = 1$)

$$r_{\pm} = \frac{2GM \pm \sqrt{(2GM)^2 - 4(\frac{J}{M})^2}}{2} \tag{2.8c}$$

The extremal Kerr black hole solution occurs when the outer and inner horizon radius coincide $r_+ = r_- = GM$. In other words, when $GM^2 = J$. Consequently the radius quantization condition reads now $(GM_n)^2 = nL_P^2 \Rightarrow \frac{M_n^2}{m_P^2} = J_n = n$, which is similar to eq-(2.1) without the factor of 4, and resembles Regge trajectories in string theory. Extremal black holes as massive string states have been analyzed by [22] and others.

However, there is a caveat now if one identifies the frequency $\omega_{n,n-1}$ in the mass transition $M_n \to M_{n-1}$ with a black hole temperature because the temperature of a *extremal* Kerr black hole is *zero*. The resolution of this problem relies on the fact that one must *not* confuse the angular momentum J of the Kerr black hole with the angular momentum **operators** $\mathbf{J}_{S^3}, \mathbf{J}_{S^2}$ associated with the (hyper) spheres $\mathbf{S}^3, \mathbf{S}^2$ in the previous section that were essential in deriving the eigenvalues and eigenfunctions of the area operators resulting from the noncommutativity of spacetime.

Case **B** : Setting $l_2 = 0$, defining $n \equiv l_3 + 1$, with $\Delta n = 1$, eqs-(1.14, 2.1) lead to the following quantization condition (omitting an irrelevant numerical factor of 4)

$$\frac{M_n^2}{m_P^2} = n^2 - 1 \Rightarrow \frac{M_n}{m_P} = \sqrt{n^2 - 1} \Rightarrow \frac{\Delta M_n}{m_P} \sim \frac{n \, \Delta n}{\sqrt{n^2 - 1}} =$$

$$\frac{\sqrt{1 + \frac{M_n^2}{m_P^2}}}{(M_n/m_P)} \Rightarrow \Delta M_n \sim m_P \sqrt{1 + \frac{m_P^2}{M_n^2}} \tag{2.9}$$

so the temperature associated to the thermal photon of frequency $\omega_{n,n-1}$ is now given by

$$\Delta M_n = \omega_{n,n-1} = T_n \sim m_P \sqrt{1 + \frac{m_P^2}{M_n^2}} \tag{2.10}$$

whose expression is associated with a temperature-mass relation of the form

$$T = T(M) = m_P \sqrt{1 + \frac{m_P^2}{M^2}} \tag{2.11}$$

and its corresponding entropy is

$$S \;=\; \int \frac{dM}{T(M)} \;=\; \int dM \; \frac{M}{m_P \; \sqrt{M^2 + m_P^2}} \;=\; \frac{1}{m_P} \; \sqrt{M^2 + m_P^2} \;=$$

$$\frac{M}{m_P} \; \sqrt{1 + \frac{m_P^2}{M^2}} \;\Rightarrow\; S_n \;=\; \frac{M_n}{m_P} \; \sqrt{1 + \frac{m_P^2}{M_n^2}} \;=\; \sqrt{n^2 - 1} \; \sqrt{1 + \frac{1}{n^2 - 1}} \;=\; n \tag{2.12}$$

after discretizing the expression for S in terms of n. One may note that when $M >> m_p \Rightarrow S \sim \frac{M}{m_P}$ so that the entropy now has the same functional behavior as the entropy of a string : *linear* in mass. Also, it is interesting that in both cases **A, B** one ends up with a discrete entropy $S_n = n$ quantized in n-bits.

Let us study case **C** : When $l_3 = l_2 + p = n + p$, with $l_2 = n$ and $p > 0$ is a positive integer. Eq-(2.1) becomes in this case (omitting an irrelevant numerical factor of 4)

$$\frac{M_n^2}{m_P^2} \;=\; (2p+1) \; l_2 \;+\; p(p+2) \;=\; A \, n \;+\; B, \;\; A \equiv 2p+1, \;\; B \equiv p(p+2) \tag{2.13}$$

and which has a Regge-like behavior $J = \alpha' M^2 + a$, with

$$J \;\leftrightarrow\; n, \;\; \alpha' \;\leftrightarrow\; \frac{1}{A \; m_P^2}, \; a \;\leftrightarrow\; -\frac{B}{A} \tag{2.14}$$

Hence, given $\Delta n = 1$, it leads to

$$\frac{M_n}{m_P} \;=\; \sqrt{A \; n + B} \;\Rightarrow\; \frac{\Delta M_n}{m_p} \;\sim\; \frac{A}{2 \; \sqrt{A \; n + B}} \;=\; \frac{A \; m_p}{2 \; M_n} \tag{2.15}$$

The frequency and temperature corresponding to the emitted thermal photon in the transition is given by

$$\Delta M_n \;=\; \omega_{n,n-1} \;=\; T_n \;\sim\; \frac{A}{2} \; \frac{m_P^2}{M_n} \tag{2.16}$$

and the expression (2.16) is associated with a temperature whose functional form in terms of the mass is

$$T \;=\; T(M) \;=\; \frac{A}{2} \; \frac{m_P^2}{M} \tag{2.17}$$

and the corresponding entropy, and its discretized form, are respectively given by

$$\int dS \;=\; \int \frac{dM}{T(M)} \;\Rightarrow\; S - S_o \;=\; \int_{M_o}^{M} dM \; \frac{2M}{A \; m_P^2} \;=\; \frac{1}{A \; m_P^2} \; (M^2 \;-\; M_o^2) \;\Rightarrow$$

$$S_n - S_o = \frac{1}{A\, m_P^2}\,(M_n^2 - M_o^2) = \frac{1}{A}\,[A\, n + B - (A\, n_o + B)] =$$

$$n - n_o \Rightarrow S_n = n \tag{2.17}$$

with $S_o \equiv S(M_o) = n_o$. Once again we find that $S_n = n$ in eq-(2.17) and the discretized entropy is again quantized in n-bits.

Degeneracy of States

We have studied above the simple cases when there is a linear relation between l_2 and l_3, or when $l_2 = 0$. As mentioned above, in the most general case, the mass quantization condition

$$\frac{4M_n^2}{m_P^2} = l_3(l_3+2) - l_2(l_2+1) = n; \quad n = 0,1,2,3\ldots \tag{2.18}$$

is related to the eigenvalues of the area operator (1.14) whose eigenfunctions $Y_{l_1 l_2 l_3}$ are described by eqs-(1.11, 1.12,1.13) in terms of three quantum numbers l_3, l_2, l_1. These solutions have a degeneracy of $2l_2+1$ associated to the different values of $l_1 = l_2, l_2-1, \ldots, 1, 0, -1, \ldots, -l_2$.

Some solutions of (2.18) are

$$l_2 = l_3 = 0,\ n = 0 \Rightarrow 2l_2+1 = 1 \tag{2.19a}$$

$$l_2 = l_3 = 1,\ n = 1 \Rightarrow 2l_2+1 = 3 \tag{2.19b}$$

$$l_2 = l_3 = 2,\ n = 2 \Rightarrow 2l_2+1 = 5 \tag{2.19c}$$

$$l_2 = 2,\ l_3 = 3,\ n = 9, \Rightarrow 2l_2+1 = 5 \tag{2.19d}$$

$$l_2 = l_3 = 9,\ n = 9 \Rightarrow 2l_2+1 = 19 \tag{2.19e}$$

From eqs-(2.19d, 2.19e) one finds two solutions ($l_2 = 2, l_3 = 3$) and ($l_2 = l_3 = 9$) leading both to $n = 9$. Their net degeneracy is 5+19 = 24 so that $d(n = 9) = 24$.

For each value of $l_3 = N$, the values of l_2 are $\{N, N-1, N-2, \ldots, 1, 0\}$, and in turn, the values of l_1 are, respectively,

$$\{N, N-1, \ldots, 1, 0, -1, \ldots, -N\}; \quad \{N-1, N-2, \ldots, 1, 0, -1, \ldots, -(N-1)\};$$

$$\{N-2, N-3, \ldots, 1, 0, -1, \ldots, -(N-2)\}; \quad \ldots \tag{2.20}$$

Therefore, given $l_3 = N$, the total number of states $Y_{l_1 l_2 l_3}$, obeying $l_3 = N \geq l_2 \geq |l_1|$ is

$$\sum_{l_3=0}^{N} \sum_{l_2=0}^{l_3} (2l_2+1) = \sum_{l_3=0}^{N} (l_3+1)^2 = \frac{(N+1)(N+2)(2N+3)}{6} \tag{2.21}$$

One must remark that $N \neq n$ except in the special cases when $l_3 = l_2 = N \Rightarrow n = N$.

Entropy in Higher Dimensions and Quantum Information

The higher-dimensional extension of the metric (1) was found by Tangherlini [11] and can be obtained by simply replacing $(d\Omega)^2 \rightarrow (d\Omega_{D-2})^2$ (the $D-2$-dim solid angle) and $1 - \frac{2GM}{r} \rightarrow 1 - (\frac{r_h}{r})^{D-3}$ where r_h is the horizon radius expressed in terms of M and the gravitational coupling G_D in D dimensions whose units are $(length)^{D-2}$. The higher dimensional metric is given by

$$ds^2 = - f(r)\,(dt)^2 + \frac{(dr)^2}{f(r)} + r^2\,(d\Omega_{D-2})^2, \;\; f(r) = 1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} r^{D-3}} \tag{2.22}$$

where G_D is the D-dim Newton's constant, M the black hole mass. The solid angle of a $D-2$-dim hypersphere is $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$. The horizon radius is determined from the condition $f(r_h) = 0$ giving

$$r_h = \left(\frac{16\pi G_D M}{(D-2)\ \Omega_{D-2}} \right)^{\frac{1}{D-3}} \tag{2.23}$$

such that the metric (3.1) can be rewritten as

$$ds^2 = -\,[\,1 - (\frac{r_h}{r})^{D-3}\,]\,(dt)^2 + [\,1 - (\frac{r_h}{r})^{D-3}\,]^{-1}\,(dr)^2 + r^2\,(d\Omega_{D-2})^2 \tag{2.24}$$

Recurring to the mathematical results on hyper-spherical harmonics [21], [8], [7] one can generalize the results of section **1** to higher dimensions so that the quantization of the radius of the horizon (a hypersphere of size $A_h = \Omega_{D-2} r_h^{D-2}$) is described by a relation of the form

$$r_h^{D-2} = (\gamma G M_n)^{\frac{D-2}{D-3}} = n\, L_P^{D-2} \;\Rightarrow\; M_n = \gamma^{-1}\, n^{\frac{D-3}{D-2}}\, \frac{L_P^{D-3}}{G} \tag{2.25}$$

with $\gamma = \frac{16\pi}{(D-2)\Omega_{D-2}}$. When $\Delta n = 1$, $G = L_P^{D-2}$, $m_P = L_P^{-1}$, in D-dimensions, one has to first approximation

$$M_n - M_{n-1} = \Delta M_n \sim m_P\, \gamma^{-1}\, \frac{D-3}{D-2}\, n^{\frac{1}{2-D}} =$$

$$m_P\, \gamma^{-1}\, \frac{D-3}{D-2}\, \frac{1}{(\gamma M_n/m_P)^{\frac{1}{D-3}}} = \gamma^{-1}\, \frac{D-3}{D-2}\, \frac{1}{(\gamma G M_n)^{\frac{1}{D-3}}} \tag{2.25}$$

When a photon of frequency $\omega_{n,n-1}$ is emitted in the $\Delta n = 1$ transition from $M_n \rightarrow M_{n-1}$, it corresponds to a temperature given by

$$\omega_{n,n-1} = T_n = \Delta M_n \sim \gamma^{-1}\, \frac{D-3}{D-2}\, \frac{1}{(\gamma G M_n)^{\frac{1}{D-3}}} \tag{2.26}$$

and one arrives at an expression for the temperature which has the very *same* functional relation (up to numerical constants) as the Hawking temperature of a Schwarzschild black hole in all dimensions $D \geq 4$ given by $T_H = \frac{D-3}{4\pi r_h}$ where r_h is the horizon radius $r_h = (\gamma G M)^{\frac{1}{D-3}}$.

Following similar steps as in the $D = 4$ case one obtains for the entropy $S = \int dM/T(M)$ an expression given by one-quarter of the horizon's hyper-area in Planck units, and the quantization of the black hole entropy $S_{BH} = \frac{\Omega_{D-2} r_h^{D-2}}{4G}$ turns out to be $S_n = n(\frac{\Omega_{D-2}}{4})$, hence up to a numerical constant the higher-dim entropy is quantized in n-bits as well. When $D = 4$ one recovers the previous result $S_n = n\pi$.

To sum up, the $(D-2)$-dim horizon is a hyper-sphere whose radius quantization condition is

$$r_n^{D-2} \ = \ (\gamma G M_n)^{\frac{D-2}{D-3}} \ = \ n \ L_P^{D-2}, \quad \gamma = \frac{16\pi}{(D-2)\Omega_{D-2}} \tag{2.27}$$

and it leads to a mass quantization of the form

$$\frac{M_n}{m_P} \ = \ \gamma^{-1} \ n^{\frac{D-3}{D-2}} \tag{2.28}$$

and which is *not* equal to a mass quantization of the form

$$\frac{M_n}{m_P} \ = \ n^{\frac{1}{D-2}} \tag{2.29}$$

except in $D = 4$, up to a numerical constant. Consequently, when $D \neq 4$, the mass quantization condition (2.28) leads to a different temperature than the temperature stemming from the mass quantization condition in (2.29). Basically this results because $m_p(GM)^{\frac{1}{D-3}} \neq (\frac{M}{m_P})^{D-3}$, except in $D = 4$. The condition (2.28) yields a temperature which has the same functional form as the Hawking black hole temperature as shown above in eq-(2.26).

Furthermore, it is not difficult to show that when one imposes a mass quantization condition of the form $A(M_n/m_P)^B = n$, with A, B constants $(B \neq 0)$ it always leads to $S_n = n$. For example, given a more general mass quantization condition of the form $(\frac{M_n}{m_P})^{D-2} = f(n)$, where $f(n)$ is a general function which is not necessarily given by a power law, one can define the continuum extension of the ratio $x_n = \frac{M_n}{m_P}$ to be $x = \frac{M}{m_P}$, and define the continuum extension of the integer n to be y. As a result, the continuum version of the general mass quantization condition is

$$x^{D-2} \ = \ f(y) \ \Rightarrow \ (D-2) \ x^{D-3} \ dx \ = \ f'(y) dy \ \Rightarrow$$

$$\frac{T}{m_P} \ = \ \frac{dx}{dy} \ = \ \frac{f'(y)}{(D-2) \ x^{D-3}} \tag{2.30}$$

where the continuum version of the temperature associated with the photon emitted in the discrete mass transitions is chosen to be defined as $T = T_P(dx/dy) = m_P(dx/dy)$

The corresponding entropy is given by the integral

$$S = \int \frac{dM}{T(M)} = \int \frac{(D-2)\ x^{D-3}\ dx}{f'(y)} = \int dy = y \Rightarrow S_n = n \quad (2.31)$$

and one finds again that the discrete entropy is quantized in n-bits as a result of the mass quantization condition. Therefore, to conclude this section we believe that some sort of "universality" principle is operating in the quantization of *mass* which leads to a quantization of entropy. In other words, given an entropy-mass relation $S = S(M)$, a mass quantization condition leads to $S = S(M_n) = n$, and one may then assign n "bits" to this discrete entropy. The implications of mass and entropy quantization in Quantum Information Theory warrants to be investigated further. For an extensive literature on the many aspects of Quantum Gravity we refer the reader to the recent encyclopedic treatise [28].

3 Mass Quantization, Counting of States, Black Hole Evaporation, Black Holes-String Correspondence

We discussed earlier how photon of frequencies

$$\omega_{n,n-1},\ \omega_{n-1,n-2},\ \omega_{n-2,n-3},\ \ldots, \omega_{1,0} \quad (3.1)$$

are emitted in the following discrete mass transitions

$$M_n \to M_{n-1}, \quad M_{n-1} \to M_{n-2}, \quad M_{n-2} \to M_{n-3}, \quad \ldots, \quad M_1 \to M_0 = 0 \quad (3.2)$$

respectively. Consequently, there are *many* possible ways in which the mass state M_n can cascade down to the ground mass state $M_0 = 0$.

For example, let us take $n = 5$. There are seven partitions of 5 [29]

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1 \quad (3.3)$$

The order-dependent partition $1+4$ is the same as $4+1$. The order-dependent partitions $1+3+1$, and $1+1+3$ are the same as $3+1+1$, and so forth. However because the mass states $M_5, M_4, M_3, M_2, M_1, M_0$ are *not* equally spaced, the transition chain $M_5 \to M_1 \to M_0$ is *not* the same as the transition chain from $M_5 \to M_4 \to M_0$, and so forth. Consequently, one must enlarge the above seven partitions (4.3) to take into account the orderings of the partitions by including the suitable permutations.

The $M_5 \to M_0$ transition involves emitting a photon of frequency $\omega_{5,0}$. The $M_5 \to M_1 \to M_0$ transition chain involves emitting a photon of frequency $\omega_{5,1}$ from $M_5 \to M_1$, followed by the emission of a photon of frequency $\omega_{1,0}$ from

$M_1 \rightarrow M_0$. Whereas the $M_5 \rightarrow M_4 \rightarrow M_0$ transition chain involves emitting a photon of frequency $\omega_{5,4}$ from $M_5 \rightarrow M_4$, followed by the emission of a photon of frequency $\omega_{4,0}$ from $M_4 \rightarrow M_0$. Because the frequencies $\{\omega_{5,1}; \omega_{1,0}\}$ are *not* the same as $\{\omega_{5,4}, \omega_{4,0}\}$, the partition $4+1$ associated with the first transition chain is *not* physically the same as the partition $1+4$ associated with the second transition chain.

In this fashion, given the mass state M_5, it can "evaporate" by cascading all the way down to the zero mass state $M_0 = 0$ in more different ways than seven. If, and only if, the process is adiabatic, the entropy will be conserved and this "evaporation" cascading process involving photons of different frequencies encodes $information$ about the entropy content associated with the mass state M_5.

If one does *not* take into account the ordering of the partitions, the partition of an integer n in the asymptotic limit $n >> 1$ admits an analytic expression of the form [29]

$$p(n) \sim \frac{1}{4n\sqrt{3}} \; e^{\pi\sqrt{\frac{2n}{3}}} \tag{3.4}$$

and the entropy associated with $p(n)$ is given by its natural logarithm

$$S = ln(p(n)) \sim \pi \sqrt{\frac{2n}{3}} - ln(4n\sqrt{3}) \tag{3.5}$$

Hence, to leading order in the large n limit, and up to a numerical coefficient, one has $S = ln(p(n)) \sim \sqrt{n} = \frac{2M_n}{m_P}$, and the entropy is *linear* in the mass M_n which has the same behavior as the string entropy.

If one wishes to obtain an entropy quadratic in the mass M, it would require to study the partition of n^2 in the asymptotic limit $n^2 >> 1$

$$p(n^2) \sim \frac{1}{4n^2\sqrt{3}} \; e^{\pi\sqrt{\frac{2n^2}{3}}} \tag{3.6}$$

such that

$$S = ln(p(n^2)) \sim \pi \sqrt{\frac{2n^2}{3}} - ln(4n^2\sqrt{3}) = n\,\pi\sqrt{\frac{2}{3}} - ln(4n^2\sqrt{3}) =$$

$$n\,\pi\sqrt{\frac{2}{3}} - 2\,ln(n) - ln(4\sqrt{3}) \tag{3.7}$$

Hence, to leading order, one has $ln(p(n^2)) \sim n\pi\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}}(\frac{4\pi M_n^2}{m_P^2}) = \sqrt{\frac{2}{3}}\frac{A_n}{4G}$, as a result of the one-quarter-area quantization condition $\frac{A_n}{4G} = n\pi$. One finds that the entropy is now *quadratic* in the mass M_n and agrees with the back hole entropy, up to a numerical constant $\sqrt{\frac{2}{3}}$. In this case, instead of studying the mass transitions among the states $M_n, M_{n-1}, M_{n-2}, \ldots, M_0 = 0$, it requires to study the mass transitions among the states $M_{n^2}, M_{n^2-1}, M_{n^2-2}, \ldots, M_0 = 0$, which are *not* the *same* as the discrete mass states $M_{n^2}, M_{(n-1)^2}, M_{(n-2)^2}, \ldots, M_0 = 0$.

Let us recall the discrete version of the entropy expression found in eq-(2.8b) : $S_n = n\pi = 4\pi(\frac{M_n}{m_P})^2$ (plus logarithmic corrections). This solution originated from the area/mass quantization condition and after performing the binomial (Taylor) expansion in powers of $(1/n)$ in eq-(2.5) and which is consistent with having a large n limit. We may compare the latter expression S_n, in the very large n limit, with the logarithm $ln[p(n^2)]$. One finds that $ln[p(n^2)]$ in eq-(3.7) leads to an expression *close*, but not exactly identical, to the black hole entropy $S_n = 4\pi(\frac{M_n}{m_P})^2$ (plus logarithmic corrections) displayed in eq-(2.8b).

If one realizes that the cascading paths from M_{n^2} to $M_0 = 0$ are actually order-dependent, because the mass states are *not* equally spaced, the number of cascading paths $\mathcal{N}$ is much *greater* than $p(n^2)$. Therefore one expects to have

$$\mathcal{N} \ > \ e^{S_n} \ \sim \ [p(n^2)] \ \Rightarrow \ ln(\mathcal{N}) > S_n \sim ln[p(n^2)] \tag{3.8}$$

in the large n limit.

Inspired by Feynman's path integral formulation of quantum mechanics (sum over paths) we may interpret the similarity of the expressions $S_n \sim ln[p(n^2)]$, in the large n limit, as an indication that the number of all possible *climbing* paths from the mass state $M_0 = 0$ to the M_{n^2} state is a measure of the entropy content of a black hole whose mass is $M_n = \frac{\sqrt{n}}{2} m_P$, and which is *not* the same as $M_{n^2} = \frac{n}{2} m_P$, except in the case when $n = 1$ (and $n = 0$ that corresponds to the zero mass state). Similarly, the number of cascading paths from the mass state M_{n^2} to the mass state $M_0 = 0$, in the large n limit, is a measure of the entropy content associated with the large number of photons of different frequencies that have been radiated in the cascading process, and which are associated to the black hole evaporation process of a mass $M_n = \frac{\sqrt{n}}{2} m_P$.

We hope that this *correspondence* picture between $S_n \leftrightarrow ln[p(n^2)]$ might provide a new glimpse into Quantum Gravity. One salient feature of this work is that the photon's frequency in the transition $M_1 = m_P \rightarrow M_0 = 0$ corresponds to the Planck temperature $\omega_{1,0} = T_P$. The salient feature of this transition leading to a zero mass, finalizing the complete black hole evaporation, is that the temperature *no* longer blows up but it reaches a "maximal" Planck temperature. A Thermal Relativity Theory based on a maximal Planck temperature was proposed in [27].

Given $M_n = \frac{\sqrt{n}}{2} m_P$, and $M_{n^2} = \frac{n}{2} m_P$ gives the relation

$$\frac{M_{n^2}}{m_P} = \ 2 \ \frac{M_n^2}{m_P^2} \tag{3.9}$$

that hints towards a black-hole/string entropy correspondence. The left hand side which is *linear* in mass has the same behavior as the entropy of a string, whereas the right hand side which is *quadratic* in mass has the same behavior as the entropy of a black hole.

The behavior of the asymptotic limit of $p(n)$ and $p(n^2)$ in eqs-(3.4,3.6) is also the same as the counting of string states (degeneracy) of a given energy E

$$d(E) \sim A\ E^{-B}\ e^{C\sqrt{E}}, \quad E \sim \infty \tag{3.10}$$

with A, B, C numerical constants. A recent study on the asymptotic density of states in solvable models of strings can be found in [26].

We finalize this section with a discussion of the black holes-string correspondence. The correspondence principle between strings and black holes is a general framework for matching black holes and massive states of fundamental strings at a point where their physical properties such as mass, entropy and temperature smoothly agree [23]. The black-hole/string correspondence [23] occurs when the string's Hagedorn temperature is of the order of the Hawking temperature. In $D = 4$, the black hole entropy is proportional to M^2, whereas the string entropy is *linear* in the mass M [23]. The string coupling $g_s = e^{<\phi>}$ is given in terms of the vacuum expectation value of the dilaton ϕ. The string mass scale M_s, the Planck mass m_P, and the mass M corresponding to a massive string state are related in the following way via the string coupling g_s [24]

$$M_s \ = \ g_s\ m_P, \ \ M_s \ = \ g_s^2\ M \ \Rightarrow \ \frac{M^2}{m_P^2} \ = \ \frac{M^2}{M_s^2}\ g_s^2 \ = \ \frac{M^2}{M_s^2}\ \frac{M_s}{M} \ = \ \frac{M}{M_s} \tag{3.11}$$

Consequently, in $D = 4$ one has $S_{BH} \sim \frac{M^2}{m_p^2} = \frac{M}{M_s} \sim S_{string}$, and one can monitor the black-hole/string transition by varying the string coupling. As explained by [24], a large black hole with a stringy stretched horizon evolves, under adiabatic change of the string coupling, to a black hole of string size, and then to a single free string. This black-hole/string correspondence in the case of rotating black holes and in higher dimensions $D > 4$ was analyzed more recently by [25]. Comparing eq-(3.9) with eq-(3.11) one can see a very clear resemblance. The only difference is that m_P appears in both ratios of eq-(3.9); whereas the ratios in eq-(3.11) involve m_P and the mass scale M_s which is the inverse of the string length scale L_s.

This is not the first time we have seen the integers n and their squared n^2 playing an important role. The energy levels of the Hydrogen atom obey the relation $E_n = -\frac{|E_0|}{n^2}$ where $E_0 = -13.6$ eV is the ground state energy. The n-th orbital velocity of the classical electron obeys the relation $v_n = \frac{v_0}{n}$ where $v_0 = \frac{c}{137}$ is the orbital velocity corresponding to the ground state. For this reason, many researchers view the Schwarzschild black hole as the "Hydrogen atom" of Quantum Gravity.

4 Matrix Models

Instead of viewing the noncommutative spacetime coordinates in terms of angular momentum operators and recurring to the eigenvalues and eigenfunctions of the area operators provided by eqs-(1.11,1.12,1.13), another approach is to consider a spacetime where the spatial coordinates become operators that are represented by $SU(N+1)$ matrices [19]. The $su(N+1)$ algebra is a subalgebra of $so(2N+2)$.

The author [20] more recently proposed a large N quantum mechanics of non-abelian bosonic and fermionic variables belonging to the adjoint representation of $SU(N+1)$ as a Matrix model for quantum gravity in $D=3$. The theory admits a fuzzy sphere of radius $R = NL_P$ as a static solution. Over the fuzzy geometry, the quantum mechanics of the fermions is given by a sum of oscillators with equal frequency. The fuzzy sphere was divided into N^2 cells with unit cell area $\Delta A = 4\pi L_P^2$ of Planck size. Each two-dim cell is populated by a pair of fermionic oscillators which describe the quantum fluctuations over the fuzzy sphere.

The energy state where exactly half of the Fermi sea is filled contains the maximal amount of degeneracy which in the large N limit turns out to be given by $\frac{2^{2N^2}}{\sqrt{\pi}N}$ [1], after using the Stirling approximation for the factorials appearing in $(2N^2)!/(N^2)!(N^2)!$. The energy of the half-filled Fermi sea turned out to be [20]

$$E \;=\; \frac{N m_P}{2} \;=\; \frac{N}{2L_P} \;=\; \frac{N L_P}{2L_P^2} \;=\; \frac{N L_P}{2G} = \frac{R}{2G} \tag{4.1}$$

This was the Schwarzschild mass-radius relation if the fuzzy sphere is identified with the black hole horizon whose radius is $R = NL_P$, and the total energy of the system is identified with the mass of the black hole $E = M$. These microstates of the system at the energy $E = M$ give rise to the entropy

$$S = log_2(\frac{2^{2N^2}}{N\sqrt{\pi}}) \;=\; 2N^2 - log_2(N\sqrt{\pi}) \;=\; 2N^2 - \frac{1}{2}log_2(N^2) - log_2(\sqrt{\pi}) \tag{4.2}$$

The leading term of (4.2) yields $2N^2 \sim \frac{A}{4G}$, and one finds that the result (4.2) has the *same* functional form (up to numerical coefficients) as the expression for the entropy displayed in eq-(3.7), obtained in the large n limit, simply by setting $n = N^2$. We should also recall the relation $log_2(X) = ln(X)/ln(2)$ when one performs the logarithm operation in different basis.

The key to the findings by [20] were based in the introduction of fermionic variables. In particular, due to the fact that each two-dim cell is populated by a pair of fermionic oscillators which describe the quantum fluctuations over the fuzzy sphere. An intuitive explanation of the findings by [20] can also be found in an Ising model involving a collection of $2N^2$ fermions of spin $j = \frac{1}{2}$. The number of spin configuration states is 2^{2N^2}, since there are two degrees of freedom, spin up/down, at each lattice site. One could engineer a model such that the lowest energy state occurs when *all* the spins are down; and the maximum energy state occurs when *all* the spins are up, and an *intermediate* energy state occurs when $half$ of the spins are up, and $half$ are down. The

[1] The factor $1/\sqrt{\pi}N$ was not included in [20]

latter intermediate energy state is the one with *maximal* degeneracy given by the binomial coefficient $C^{2N^2}_{N^2} = (2N^2)!/(N^2)!(N^2)!$, and one recovers the same result as in [20], which is not surprising since similar results have been found in the evaluation of black hole entropy in Loop Quantum Gravity via Penrose spin networks.

Therefore, one concludes that the large N Matrix model (fuzzy sphere) approach of [20] leads to similar results for the black hole entropy as the model described in this work which is based on the discrete mass transitions of $M_{n^2}, M_{n^2-1}, M_{n^2-2}, \ldots, M_0 = 0$, in the large n limit, and originating from the *noncommutativity* of the spacetime coordinates which resulted in the quantization of area and mass.

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