WEIGHTED $m$-GENERALIZED GROUP INVERSE IN $\ast$-BANACH ALGEBRAS

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Abstract. Recently, Gao, Zuo and Wang introduced the $W$-weighted $m$-weak group inverse for complex matrices which generalized the (weighted) core-EP inverse and the WC inverse. The main purpose of this paper is to extend the concept of $W$-weighted $m$-weak group inverse for complex matrices to elements in a Banach $\ast$-algebra. This extension is called $w$-weighted $m$-generalized group inverse. We present various properties, presentations of such new weighted generalized inverse. Related (weighted) $m$-generalized core inverses are investigated as well. Many properties of the $W$-weighted $m$-weak group inverse are thereby extended to wider cases.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$  

Such $x$ is unique if exists, denoted by $a^\#$, and called the group inverse of $a$ (see [14]). As is well known, a square complex matrix $A$ has group inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$.

A Banach algebra is called a Banach $\ast$-algebra if there exists an involution $\ast : x \to x^\ast$ satisfying $(x + y)^\ast = x^\ast + y^\ast, (\lambda x)^\ast = \overline{\lambda} x^\ast, (xy)^\ast = y^\ast x^\ast, (x^\ast)^\ast = x$. The involution $\ast$ is proper if $x^\ast x = 0 \implies x = 0$ for any $x \in \mathcal{A}$, e.g., in a Rickart $\ast$-algebra, the involution is always proper. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $\ast$ as the involution. Then the involution $\ast$ is proper. In [21], Zou et al. extended the notion of weak group inverse from complex matrices to elements in a ring with proper involution.

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Let $\mathcal{A}$ be a Banach algebra with a proper involution $\ast$. An element $a$ in $\mathcal{A}$ has weak group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, a^n = xa^{n+1}$$

for some $n \in \mathbb{N}$. Such $x$ is unique if it exists and is called the weak group inverse of $a$. We denote it by $a^W$ (see [21, 22]). A square complex matrix $A$ has weak group inverse $X$ if it satisfies the system of equations:

$$AX^2 = X, AX = \mathcal{A}^\mathcal{D}A.$$ 

Here, $\mathcal{A}^\mathcal{D}$ is the core-EP inverse of $A$ (see [11, 23]). Weak group inverse was extensively studied by many authors, e.g., [8, 17, 20, 21, 22].

In [2], the authors extended weak group inverse and introduced generalized group inverse in a Banach algebra with proper involution. An element $a$ in $\mathcal{A}$ has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n \to \infty} \|a^n - xa^{n+1}\|^\frac{1}{n} = 0.$$

Such $x$ is unique if it exists and is called the generalized group inverse of $a$. We denote it by $a^\mathcal{G}$. Many properties of generalized group inverse were presented in [2]. Mosić and Zhang introduced and studied weighted weak group inverse for a Hilbert space operator $A$ in $\mathcal{B}(X)$ (see [17]). Furthermore, the weak group inverse was generalized to the $m$-weak group inverse (see [11, 18, 24]). Recently, Gao et al. further introduced and studied the $W$-weighted $m$-weak group inverse in [11].

The main purpose of this paper is to extend the concept of $W$-weighted $m$-weak group inverse for complex matrices to elements in a Banach *-algebra. This extension is called weighted $m$-generalized group inverse.

An element $a \in \mathcal{A}$ has generalized $w$-Drazin inverse $x$ if there exists unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \quad \text{and} \quad a - awxwa \in \mathcal{A}^{qnil}.$$ 

We denote $x$ by $a^{d,w}$ (see [19]). Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n \to \infty} \|x^n\|^\frac{1}{n} = 0\}$. We denote $a^{d,1}$ by $a^d$. Evidently, $a^{d,w} = x$ if and only if $x = a[(wa)^d]^2$. We introduce a new weighted generalized inverse as follows:

**Definition 1.1.** An element $a \in \mathcal{A}$ has $w$-weighted $m$-generalized group inverse if $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*wa,$$

$$\lim_{n \to \infty} \|(aw)^n - (xw)(aw)^{n+1}\|^\frac{1}{n} = 0.$$
The preceding $x$ is called the $w$-weighted $m$-generalized group inverse of $a$, and denoted by $a^{\ominus_m,w}$.

The $w$-weighted $m$-generalized group inverse is a natural generalization of the $m$-generalized group inverse which was introduced in [1]. Let $a^{\ominus_m}$ be the $m$-generalized group inverse of $a$. Evidently, $a^{\ominus_m} = a^{\ominus_{m+1}}$. We list some characterizations of $m$-generalized group inverse.

**Theorem 1.2.** (see [1, Theorem 2.3, Theorem 3.1 and Theorem 4.1]) Let $\mathcal{A}$ be a Banach $*$-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:

1. $a \in \mathcal{A}^{\ominus_m}$.
2. There exist $x, y \in \mathcal{A}$ such that
   \[ a = x + y, x^*a^{m-1}y = yx = 0, x \in \mathcal{A}^\#, y \in \mathcal{A}^{qnil}. \]
3. $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that
   \[ x = ax^2, (a^d)^*a^{m+1}x = (a^d)^*a^m, \lim_{n \to \infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0. \]
4. $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that
   \[ x = ax^2, (a^d)^*a^{m+1}x = (a^d)^*a^m, \lim_{n \to \infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0. \]
5. $a \in \mathcal{A}^d$ and there exists an idempotent $p \in \mathcal{A}$ such that
   \[ a + p \in \mathcal{A}^{-1}, [(a^m)^*a^m p]^{*} = a^{*}ap \text{ and } pa = pap \in \mathcal{A}^{qnil}. \]
6. $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that $(a^d)^*a^d x = (a^d)^*a^m$.

In Section 2, we investigate elementary properties of $w$-weighted $m$-generalized group inverse in a Banach $*$-algebra. Many new properties of the weak group inverse for a complex matrix and Hilbert space operator are thereby obtained.

Following [3], an element $a$ in $\mathcal{A}$ has generalized $w$-core-EP inverse if there exist $x \in \mathcal{A}$ such that
\[
a(wx)^2 = x, (wawx)^* = wawx, \lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.\]

The preceding $x$ is unique if exists, and we denote it by $a^{\oplus,w}$. We denote $a^{\oplus,1}$ by $a^{\oplus}$. Evidently, $a^{\oplus,w} = x$ if and only if $x = a[(wa)^{\oplus}]^2$ (see [3, Theorem 2.1]). In Section 3, we investigate the representations of $m$-generalized group inverse under weighted generalized core-EP invertibility.

Recall that an element $a \in \mathcal{A}$ has Moore-Penrose inverse if there exist $x \in \mathcal{A}$ such that $axa = a, xax = x, (ax)^* = ax, (xa)^* = xa$. The preceding $x$ is unique if it exists, and we denote it by $a^\dagger$. An element $a$ in $\mathcal{A}$ has weak core inverse provided that $a \in \mathcal{A}^W \cap \mathcal{A}^\dagger$ (see [16, 23]). In [4], the authors introduced and
studied the generalized core inverse. The \( m \)-weak core inverse and weighted weak core inverse were investigated in [10, 15]. Recently, Ferreyra and Mosić introduced the \( W \)-weighted \( m \)-weak core inverse for complex matrices which generalized the (weighted) core-EP inverse, the weak group inverse and \( m \)-weak core inverse (see [7]). A square complex matrix \( A \) has \( W \)-weighted \( m \)-weak core inverse \( X \) if

\[
X = A^{\mathbb{W},m,W}(WA)^m[(WA)^m]^\dagger.
\]

Here, \( A^{\mathbb{W},m,W} \) is the \( W \)-weighted \( m \)-weak group inverse of \( A \), i.e., \( (WA)^m \) has weak group inverse (see [20]). Let \( a, w \in \mathcal{A}, m \in \mathbb{N} \). Set \( a \in \mathcal{A}^{\mathbb{d},m,w} \) if \( (wa)^m \in \mathcal{A}^{\dagger} \). We have

**Definition 1.3.** An element \( a \in \mathcal{A} \) has \( w \)-weighted \( m \)-generalized core inverse if \( a \in \mathcal{A}^{m,w} \cap \mathcal{A}^{\mathbb{d},m,w} \).

In Section 4, We present various properties, presentations of such weighted generalized group inverse combined with weighted Moore-Penrose inverse. We extend the properties of generalized core inverse in Banach *-algebra to the general case (see [4]). Many properties of the \( W \)-weighted \( m \)-weak core inverse are thereby extended to wider cases, e.g. Hilbert operators over an infinitely dimensional space.

Finally, in Section 5, we give the applications of the \( w \)-weighted \( m \)-generalized group (core) inverse in solving the matrix equations.

Throughout the paper, all Banach algebras are complex with a proper involution \( * \). We use \( \mathcal{A}^{\dagger}, \mathcal{A}^{\mathbb{d},w}, \mathcal{A}^{\mathbb{g}} \) and \( \mathcal{A}^{\mathbb{W}} \) to denote the sets of all Moore-Penrose invertible, weighted generalized Drazin invertible, generalized core-EP invertible, generalized group invertible and weak group invertible elements in \( \mathcal{A} \), respectively.

### 2. WEIGHTED \( m \)-GENERALIZED GROUP INVERSE

In this section we introduce and establish elementary properties of weighted \( m \)-generalized group inverse which will be used in the next section. This also extend the concept of \( w \)-weighted \( m \)-weak group inverse from complex matrices to elements in a Banach algebra (see [11]). We begin with

**Theorem 2.1.** Let \( a, w \in \mathcal{A} \). Then the following are equivalent:

1. \( a \in \mathcal{A}^{\mathbb{g},m,w} \).
2. \( wa \in \mathcal{A}^{\mathbb{g},m} \).

In this case, \( a^{\mathbb{g},m,w} = a[(wa)^{\mathbb{g}}]_m^2 \).
Proof. (1) $\Rightarrow$ (2) By hypothesis, we can find $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*wa,$$

$$\lim_{n \to \infty} \|(aw)^{n-1} - (xw)(aw)^{n}\|^{\frac{1}{n}} = 0.$$ 

Furthermore, we have

$$\|(wa)^{n} - (wx)(wa)^{n+1}\|^{\frac{1}{n}}$$

$$= \|w(aw)^{n-1}a - wxw(aw)^{n}a\|^{\frac{1}{n}}$$

$$\leq \|w\|^\frac{1}{n} \|[aw(aw)^{n-1} - xw(aw)^{n}]a\|^{\frac{1}{n-1}} \|a\|^{\frac{1}{n}}.$$ 

Therefore

$$\lim_{n \to \infty} \|(wa)^{n} - (wx)(wa)^{n+1}\|^{\frac{1}{n}} = 0.$$ 

Obviously, $wx = (wa)(wx)^2$. Hence,

$$wa \in \mathcal{A}^\otimes_m$$

and $(wa)^\otimes_m = wx$.

Accordingly,

$$x = a(wx)^2 = a[(wa)^\otimes_m]^2,$$

as desired.

(2) $\Rightarrow$ (1) Let $x = a[(wa)^\otimes_m]^2$. Then $a \in \mathcal{A}^{d,w}$ and we verify that

$$a(wx)^2 = a[(wa)^\otimes_m]^2wa[(wa)^\otimes_m]^2$$

$$= a[(wa)^\otimes_m]^2 = x.$$ 

One easily checks that

$$[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^{m+1}wa[(wa)^\otimes_m]^2$$

$$= [(wa)^d]^*(wa)^{m+1}(wa)^\otimes_m$$

$$= [(wa)^d]^*wa.$$ 

Since

$$(xw)(aw)^{n+1} = a[(wa)^\otimes_m]^2w(aw)^{n+1}$$

$$= (aw)^n - a[(wa)^\otimes_m]^2(wa)^{n+1}$$

$$- a(aw)^\otimes_m[(wa)^{n} - (wa)^\otimes_m(aw)^{n+1}]w,$$

we have

$$\|\|(aw)^{n} - (xw)(aw)^{n+1}\|^{\frac{1}{n}}$$

$$\leq \|a\|^\frac{1}{n} \|[aw(aw)^{n-1} - (wa)^\otimes_m(aw)^{n}]a\|^{\frac{1}{n-1}} \|a\|^\frac{1}{n}$$

$$+ \|a(aw)^\otimes_m\|^\frac{1}{n} \|[aw)^{n} - (wa)^\otimes_m(aw)^{n+1}]\|^{\frac{1}{n}} \|w\|^\frac{1}{n}.$$ 

Therefore

$$\lim_{n \to \infty} \|(aw)^{n} - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0,$$
the result follows.

The preceding unique solution $x$ is called the $w$-weighted generalized $m$-group inverse of $a$, and denote it by $a^{\otimes_m,w}$. That is, $a^{\otimes_m,w} = a[(wa)^{\otimes_m}]^2$. We use $\mathcal{A}^{\otimes_m,w}$ to denote the set of all $w$-weighted generalized $m$-group invertible elements in $\mathcal{A}$. By the argument above, we have

**Corollary 2.2.** Let $a, w \in \mathcal{A}$. Then

1. $a^{\otimes_m,w} = x$.
2. $wa \in \mathcal{A}^{\otimes_m}$ and $(wa)^{\otimes_m} = wx$.

**Corollary 2.3.** Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\otimes_m,w}$ if and only if

1. $a \in \mathcal{A}^{d,w}$;
2. There exists $x \in \mathcal{A}$ such that
   \[
   x = a[wx]^2, [(wa)^* (wa)^{m+1}wx]^* = (wa)^* (wa)^{m+1}wx, \\
   \lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
   \]

**Proof.** $\Rightarrow$ Obviously, $a \in \mathcal{A}^{d,w}$. By hypothesis, there exists $x \in \mathcal{A}$ such that

\[
(xw)(aw)^{n+1} = [(wa)^{d} (wa)^{m+1}wx]^*\]

In this case, $x = a[(wa)^{\otimes_m}]^2$. Then

\[
(wa)^* (wa)^{m+1}wx = (wa)^* (wa)^{m+1}wa[(wa)^{\otimes_m}]^2 \\
= (wa)^* (wa)^{m+1}(wa)^{\otimes_m}, \\
((wa)^* (wa)^{m+1}wx)^* = (wa)^* (wa)^{m+1}wx.
\]

$\Leftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that

\[
x = a[wx]^2, [(wa)^* (wa)^{m+1}wx]^* = (wa)^* (wa)^{m+1}wx, \\
\lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]

Clearly, $wx = (wa)[wx]^2$. Observing that

\[
||((wa)^{n+1} - (wx)(wa)^{n+2})|| \\ 
\leq ||w|| ||(aw)^n - (w(xw)(aw)^{n+1}a)|| \\
\leq ||w|| ||(aw)^n - (xw)(aw)^{n+1}|| ||a||,
\]

we see that

\[
\lim_{n \to \infty} ||(aw)^n - (wx)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]

This implies that $wa \in \mathcal{A}^{\otimes_m}$. According to Theorem 2.1, $a \in \mathcal{A}^{\otimes_m,w}$, as asserted.

**Theorem 2.4.** Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\otimes_m,w}$ if and only if
(1) $a \in \mathcal{A}^{d,w}$;
(2) There exists $x \in \mathcal{A}$ such that
\[
x = a[wx]^2, [(wa)^m]^*(wa)^{m+1}wx]^* = ((wa)^m)^*(wa)^{m+1}wx,
\]
\[
\lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]

**Proof.** Clearly, $a \in \mathcal{A}^{d,w}$. In view of Theorem 2.1, $wa \in \mathcal{A}^\oplus_m$. According to Theorem 1.2, there exists $z \in \mathcal{A}$ such that
\[
z = (wa)z^2, [(wa)^m]^*(wa)^{m+1}z]^* = ((wa)^m)^*(wa)^{m+1}z,
\]
\[
\lim_{n \to \infty} ||(wa)^n - z(wa)^{n+1}||^{\frac{1}{n}} = 0.
\]
Here, $z = (wa)^\oplus_m = wa[(wa)^\oplus_m]^2$. Set $x = a[(wa)^\oplus_m]^2$. Then
\[
[(wa)^m]^*(wa)^{m+1}wz]^* = ((wa)^m)^*(wa)^{m+1}wz,
\]
\[
\lim_{n \to \infty} ||(aw)^n - (zw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]
Moreover, we have
\[
wx = wa[(wa)^\oplus_m]^2 = (wa)^\oplus_m,
\]
and then
\[
a(wx)^2 = a[(wa)^\oplus_m]^2 = x.
\]
In this case, $a \oplus_m w = x$, as desired.

$\Leftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that
\[
x = a(wx)^2, [(wa)^m]^*(wa)^{m+1}wx]^* = ((wa)^m)^*(wa)^{m+1}wx,
\]
\[
\lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]
Then $wx = wa(wx)^2$. In view of Theorem 1.2, $wa \in \mathcal{A}^\oplus_m$. According to Theorem 2.1, $a \in \mathcal{A}^\oplus_m w$, as asserted. \hfill $\square$

**Corollary 2.5.** Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^\oplus_m w$ if and only if
(1) $a \in \mathcal{A}^{D,w}$;
(2) There exists $x \in \mathcal{A}$ such that
\[
x = a[wx]^2, [(wa)^m]^*(wa)^2wx]^* = (wa)^m*(wa)^2wx,
\]
\[
\lim_{n \to \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.
\]

**Proof.** This is obvious by Theorem 2.4. \hfill $\square$

Set $im(x) = \{xr \mid r \in \mathcal{A}\}$. We are ready to prove:

**Theorem 2.6.** Let $a, w \in \mathcal{A}$. Then the following are equivalent:
(1) $a \oplus_m w = x$. 

Proof. This is obvious by Theorem 2.4. \hfill $\square$
(2) \( ax = (wa)^{\circ}_m, awx^2 = x. \)
(3) \( wa(wa)^{\circ}_m, \text{im}(x) \subseteq \text{im}(aw)^d. \)
(4) \( awx = (wa)^{\circ}_m, \text{im}(x) \subseteq \text{im}(aw)^d. \)

**Proof.** (1) ⇒ (2) In view of Theorem 2.1, \( x = (wa)^{\circ}_m. \) Then
\[
awx = \begin{cases} 
   (aw)a[(wa)^{\circ}_m]^2 \\
   = a[(wa)(wa)^{\circ}_m]^2 \\
   = a(wa)^{\circ}_m.
\end{cases}
\]

\[\]

(2) ⇒ (3) Obviously, \( wawx = w(ax) = w(a(wa)^{\circ}_m) = wa(wa)^{\circ}_m. \) Moreover, we have
\[
x = a(wx)^2 = awxwx = a(wa)^{\circ}_mwx \\
   = a(wa)^d(wa)(wa)^{\circ}_mwx \\
   = a[(wa)^d]^2w(awa)(wa)^{\circ}_mwx \\
   = (aw)^d(a(awa)(wa)^{\circ}_mwx.
\]

Therefore \( \text{im}(x) \subseteq \text{im}(aw)^d, \) as desired.

(3) ⇒ (4) Since \( \text{im}(x) \subseteq \text{im}(aw)^d, \) we see that
\[\]
\[
awx = \begin{cases} 
   a[(aw)^d]^2w(awa)(wa)^{\circ}_mwx \\
   = (aw)^d(a(awa)(wa)^{\circ}_mwx \\
   = a(wa)(wa)^{\circ}_m \\
   = a(wa)^{\circ}_m,
\end{cases}
\]
\[\]
as desired.

(4) ⇒ (1) Write \( x = (aw)^dz \) for some \( z \in R. \) Then
\[
x = aw(ax)^d = (aw)^d(ax) \\
   = (aw)^d[a(awa)^{\circ}_m] \\
   = (aw)^d[(wa)^d]^2[(wa)^{\circ}_m] \\
   = a[(wa)^{\circ}_m]^2.
\]
This completes the proof by Theorem 2.1.

**Corollary 2.7.** Let \( a \in A. \) Then the following are equivalent:

(1) \( a^{\circ}_m = x. \)
(2) \( ax = a(a^{\circ}_m, ax^2 = x. \)
(3) \( ax = a(a^{\circ}_m, \text{im}(x) \subseteq \text{im}(a)^d. \)

**Proof.** This is a direct consequence of Theorem 2.6. □

We are ready to prove:
Theorem 2.8. Let \( a \in \mathcal{A}^{\otimes_m,w} \). Then \( wawa^{\otimes_{m+1}} = wa^{\otimes_m,wa} \).

Proof. In view of Theorem 2.1, we see that
\[
\begin{align*}
wawa^{\otimes_{m+1},w} &= wawa[(wa)^{\otimes_{m+1}}]^2 \\
&= wa(wa)^{\otimes_{m+1}} \\
wa^{\otimes_{m,wa}} &= wa[(wa)^{\otimes_m}]^2wa \\
&= (wa)^{\otimes_{m,wa}}.
\end{align*}
\]
In view of [1, Corollary 2.4], we have
\[
(wa)^{\otimes_{m+1}} = [(wa)^{\otimes_m}]^2wa.
\]
Therefore
\[
\begin{align*}
wawa^{\otimes_{m+1},w} &= wa(wa)^{\otimes_{m+1}} \\
&= wa[(wa)^{\otimes_m}]^2wa \\
&= (wa)^{\otimes_{m,wa}} \\
&= wa^{\otimes_{m,wa}}.
\end{align*}
\]
This completes the proof. \(\square\)

Corollary 2.9. Let \( a \in \mathcal{A}^{\otimes_m} \). Then \( aa^{\otimes_{m+1}} = a^{\otimes_{m+1}}a \).

Proof. This is obvious by choosing \( w = 1 \) in Theorem 2.8. \(\square\)

3. REPRESENTATIONS OF \( m \)-GENERALIZED GROUP INVERSE

In this section, we present the representations of \( m \)-generalized group inverse under weighted generalized core-EP invertibility.

Theorem 3.1. Let \( a \in \mathcal{A}^{\ominus,w} \). Then \( a \in \mathcal{A}^{\otimes_{m,w}} \) and
\[
a^{\otimes_{m,w}} = [a^{\ominus,w}w]^{m+1}(aw)^{m-1}a.
\]

Proof. In view of [3, Theorem 2.1], \( a^{\ominus,w} = a[(wa)^{\ominus}]^2 \); hence, \( wa^{\ominus,w} = (wa)^{\ominus} \). Then we easily check that
\[
\begin{align*}
[a^{\ominus,w}w]^{m+1}(aw)^{m-1}a &= a^{\ominus,w}[wa^{\ominus,w}]^m(aw)^{m-1}a \\
&= a^{\ominus,w}[(wa)^{\ominus}]^m(aw)^m \\
&= a[(wa)^{\ominus}]^2[(wa)^{\ominus}]^m(aw)^m \\
&= a[(wa)^{\ominus}]^{m+2}(wa)^m.
\end{align*}
\]
Thus,
\[w[a^{\ominus,w}w]^{m+1}(aw)^{m-1}a = wa[(wa)^{\ominus}]^{m+2}(wa)^m = [(wa)^{\ominus}]^{m+1}(wa)^m.\]
Set \( x = [(wa)^{\ominus}]^{m+1}(wa)^m \). Then
According to Theorem 2.1, we prove that $\lim_{n \to \infty} \frac{||((wa)^n - x(wa)^{n+1})||^\frac{1}{n}}{n} = 0.$

This implies that $\left((wa)^\oplus \right)^m = [(wa)^\oplus]^{m+1}(wa)^m$.

According to Theorem 2.1, we prove that $a \in \mathcal{A}^{\otimes_m,w}$ and

\[
\begin{align*}
(a^\otimes_m)^w &= a[(wa)^\otimes_m]^2 \\
&= a[(wa)^\otimes_m]^{m+1}(wa)^m \\
&= a[(wa)^\otimes_m]^{m+1}(wa)^m \\
&= a[(wa)^\otimes_m]^{m+1}(wa)^m \\
&= a[(wa)^\otimes_m]^{m+1}(wa)^m \\
&= a[(wa)^\otimes_m]^{m+1}(wa)^m \\
&= [a^\otimes_m (wa)]^{m+1}(aw)^{m-1}a,
\end{align*}
\]

as required. \hfill \Box

**Corollary 3.2.** Let $a \in \mathcal{A}^{\otimes}$. Then $a \in \mathcal{A}^{\otimes_m}$ and

\[
a^{\otimes_m} = (a^{\otimes})^{m+1} a^m.
\]

**Proof.** This is obvious by choosing $w = 1$ in Theorem 3.1. \hfill \Box

We call $x$ is the $(1,3)$-inverse of $a$ if $x$ satisfies the equations $axa = a$ and $(ax)^* = ax$. We use $\mathcal{A}^{(1,3)}$ to denote the set of all $(1,3)$-invertible elements in $\mathcal{A}$. Let $a \in \mathcal{A}^{\otimes,w}$ and $a(wa)^\otimes w \in \mathcal{A}^{(1,3)}$. By using [3, Theorem 2.5], $aw, wa \in \mathcal{A}^{\otimes}$. Let $p = (aw)(aw)^\otimes, q = (wa)(wa)^\otimes$. Then $p,q \in \mathcal{A}$ are projections.

**Lemma 3.3.** Let $a \in \mathcal{A}^{\otimes,w}$ and $a(wa)^\otimes w \in \mathcal{A}^{(1,3)}$. Then

\[
a = \left( \begin{array}{cc} a_1 & a_2 \\ 0 & a_3 \end{array} \right)_{p,q}, \quad w = \left( \begin{array}{cc} w_1 & w_2 \\ 0 & w_3 \end{array} \right)_{q,p},
\]

where $a_1 \in [pAq]^{-1}, w_1 \in [qAp]^{-1}$ and $a_3w_3$ and $w_3a_3$ are quasinipotent.
\textbf{Proof.} We easily verify that
\begin{align*}
(1-p)aq &= [1-(aw)(aw)\oplus]a(wa)(wa)\oplus \\
&= [1-(aw)(aw)\oplus]awa(wa)^n[(wa)\oplus]^{n+1} \\
&= [1-(aw)(aw)\oplus](aw)^{n+1}a[(wa)\oplus]^{n+1} \\
&= aw[(aw)^n-(aw)\oplus]^{n+1}a[(wa)\oplus]^{n+1}.
\end{align*}
Then
\[\|\|(1-p)aq\|\|^{\frac{1}{\lambda}} \leq \|aw\|^{\frac{1}{\lambda}}\|\|(aw)^n-(aw)\oplus]^{n+1}\|^{\frac{1}{\lambda}}\|a[(wa)\oplus]^{n+1}\|^{\frac{1}{\lambda}}.\]
Since \(\lim_{n \to \infty} \|\|(aw)^n-(aw)\oplus]^{n+1}\|^{\frac{1}{\lambda}} = 0\), we see that \(\lim_{n \to \infty} \|\|(1-p)aq\|\|^{\frac{1}{\lambda}} = 0\).
This implies that \((1-p)aq = 0\). Likewise, we prove that
\[(1-q)wp = [1-(wa)(wa)\oplus]w(aw)(aw)\oplus = 0.\]
Moreover, we have
\begin{align*}
[(aw)(aw)\oplus]a(wa)(wa)\oplus &\quad [(wa)(wa)\oplus]w(aw)(aw)\oplus \\
&= (aw)(aw)\oplus a(wa)(wa)\oplus w(aw)\oplus w(aw)\oplus \\
&= (aw)(wa)\oplus w(aw)\oplus a(aw)(wa)\oplus \\
&= (wa)(wa)\oplus w(aw)\oplus a(wa)(wa)\oplus \\
&= (wa)(wa)\oplus w(a)\oplus a(wa)\oplus \\
&= (wa)(wa)\oplus a(wa)\oplus \\
&= (wa)(wa)\oplus.
\end{align*}
Then \(a_1 = paq \in [pAp]^{-1}\). Similarly, \(w_1 = qwp \in [qAp]^{-1}\).
Also we easily see that
\[a_3w_3 = [1-(aw)(aw)\oplus]a[1-wa(wa)\oplus]w[1-(aw)(aw)\oplus] \in A^{nil}.\]
Thus, \(a_3w_3\) is quasinilpotent. By using Cline’s formula, \(w_3a_3\) is quasinilpotent. This completes the proof. \(\square\)

\textbf{Lemma 3.4.} Let \(a \in A^{\oplus,w}\) and \((wa)\oplus w \in A^{(1,3)}\). Then
\[a^{\oplus,w} = \begin{pmatrix}
(w_1a_1w_1)^{-1} & 0 \\
0 & 0
\end{pmatrix}_{p,q}.
\]
Proof. In view of [3, Theorem 2.1], $a_{\otimes,w} = a[(wa)_{\otimes}]^2$. One easily checks that

\[
p a_{\otimes,w} (1 - q) = (aw)(aw)_{\otimes}a[(wa)_{\otimes}]^2[1 - (wa)(wa)_{\otimes}] \\
= (aw)(aw)_{\otimes}a[(wa)_{\otimes}]^3[(wa)_{\otimes} - (wa)_{\otimes}(wa)(wa)_{\otimes}] = 0,
\]

\[
(1 - p)a_{\otimes,w} q = [1 - (aw)(aw)_{\otimes}]a[(wa)_{\otimes}]^2(wa)(wa)_{\otimes} \\
= [1 - (aw)(aw)_{\otimes}]awa[(wa)_{\otimes}]^3(wa)(wa)_{\otimes} \\
= aw[1 - (aw)_{\otimes}aw][wa][wa]_{\otimes}^3(wa)(wa)_{\otimes} = 0,
\]

\[
(1 - p)a_{\otimes,w} (1 - q) = [1 - (aw)(aw)_{\otimes}]a[(wa)_{\otimes}]^2[1 - (wa)(wa)_{\otimes}] = 0.
\]

Moreover, we see that

\[
p a_{\otimes,w} q = (aw)(aw)_{\otimes}a[(wa)_{\otimes}]^2(wa)(wa)_{\otimes} \\
= (aw)(aw)_{\otimes}a[(wa)_{\otimes}]^2 \\
= a[(wa)_{\otimes}]^2 \\
= w_1a_1w_1 \in (pAQ)^{-1},
\]

thus yielding the result. \qed

**Theorem 3.5.** Let $a \in A_{\otimes,w}$ and $a(wa)_{\otimes} \in A_{(1,3)}$. Then

\[
a_{\otimes,w} = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}_{p,q},
\]

where

\[
\alpha = (w_1a_1w_1)^{-1}, \quad \beta = (w_1a_1w_1)^{-1}a_2 + [(w_1a_1w_1)^{-1}w_1][m+1]c_{n-1}a_3 + b_{m+1}(a_3w_3)^{m-1}a_3; \\
b_1 = (w_1a_1w_1)^{-1}w_2, b_{n+1} = (w_1a_1w_1)^{-1}w_1b_n, \\
c_1 = a_1w_2 + a_2w_3, c_{n+1} = a_1w_1c_n + (a_1w_2 + a_2w_3)(a_3w_3)^m.
\]

Proof. Construct two series $\{b_n\}$ and $\{c_n\}$ by the equalities: Here,

\[
b_1 = (w_1a_1w_1)^{-1}w_2, b_{n+1} = (w_1a_1w_1)^{-1}w_1b_n, \\
c_1 = a_1w_2 + a_2w_3, c_{n+1} = a_1w_1c_n + (a_1w_2 + a_2w_3)(a_3w_3)^m.
\]

Then we compute that

\[
\left[ \begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} \right]^{m+1} = \left[ \begin{pmatrix} (w_1a_1w_1)^{-1}w_1 & b_{m+1} \\ 0 & 0 \end{pmatrix} \right], \\
\left[ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} \right]^{m+1} = \left[ \begin{pmatrix} (a_1w_1)^{m-1} & c_{m-1} \\ 0 & (a_3w_3)^{m-1} \end{pmatrix} \right].
\]
According to Theorem 3.1 and Lemma 3.4, we derive

\[ a^@m,w = [a^@w]^{m+1}(aw)^{m-1}a \]

\[ = \left( \begin{pmatrix} (w_1a_1w_1)^{-1} & b_{m+1} \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} (a_1w_1)^{m-1} & c_{m-1} \\ 0 & (a_3w_3)^{m-1} \end{pmatrix} \right) \left( \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \right) \]

where

\[ \alpha = (w_1a_1w_1)^{-1}, \]
\[ \beta = (w_1a_1w_1)^{-1}a_2 + [(w_1a_1w_1)^{-1}w_1]^{m+1}c_{m-1}a_3 + b_{m+1}(a_3w_3)^{m-1}a_3. \]

This completes the proof. \( \square \)

**Corollary 3.6.** Let \( a \in A^@. \) Then

\[ a^@m = \begin{pmatrix} a_1^{-1} & (a_1)^{-1}(m+1)b_m \\ 0 & 0 \end{pmatrix}_{s,t}, \]

where \( b_1 = a_2, b_{m+1} = a_1b_m + a_2a_3^m, s = aa^@ \) and \( t = a^@a. \)

**Proof.** This is immediate by choosing \( w = 1 \) in Theorem 3.5. \( \square \)

**4. Weighted \( m \)-Generalized Core Inverse**

The aim of this section is to investigate weighted \( m \)-generalized group inverse with weighted Moore-Penrose inverse. We introduce and study weighted \( m \)-generalized core inverse in a Banach \( * \)-algebra. Let \( p_{(wa)}^m = (wa)^m[(wa)^m]^\dagger \) be the projection on \( (wa)^m \). The following theorem is crucial.

**Theorem 4.1.** Let \( a \in A^@_{m,w}. \) Then there exists a unique \( x \in A \) such that

\[ xwawx = x, awx = awa^@_{m,w}p_{(wa)}^m, x(\langle wa \rangle)^m = a^@_{m,w}(\langle wa \rangle)^m. \]

**Proof.** Taking \( x = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger. \) Then

\[ xwawx = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger awa^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger = x. \]

\[ awx = awa^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger = awa[(\langle wa \rangle)^2(\langle wa \rangle)^m]^\dagger = a(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger, \]
\[ x(\langle wa \rangle)^m = a^@_{m,w}(\langle wa \rangle)^m[(\langle wa \rangle)^m]^\dagger(\langle wa \rangle)^m = a^@_{m,w}(\langle wa \rangle)^m. \]
Suppose that \( x' \) satisfies the preceding equations. Then one checks that

\[
\begin{align*}
x' &= x' w a w x' = a^{\otimes m,w} w a w x' \\
&= a^{\otimes m,w} w a w a^{\otimes m,w} (w a)^m [(w a)^m]^\dagger \\
&= a^{\otimes m,w} w a w a^{\otimes m,w} [(w a)^m]^2 [(w a)^m]^\dagger \\
&= a[(w a)^{\otimes m,w}]^2 w a (w a)^{\otimes m,w} [(w a)^m]^\dagger \\
&= a[(w a)^{\otimes m,w}]^2 [(w a)^m]^2 [(w a)^m]^\dagger \\
&= a^{\otimes m,w} (w a)^m [(w a)^m]^\dagger \\
&= x,
\end{align*}
\]

as required. \( \square \)

We denote the preceding unique \( x \) by \( a^{\otimes m,w} \).

**Corollary 4.2.** Let \( a \in A^{\otimes m,w}(m \geq 2) \). Then the following are equivalent:

1. \( a^{\otimes m,w} = x \).
2. The equation system

\[
awx = a(wa)^{\otimes m,p(wa)^m}, a(wx)^2 = x
\]

is consistent and its unique solution \( x = a^{\otimes m,w} \).

**Proof.** (1) \( \Rightarrow \) (2) In view of Theorem 4.1, we have

\[
\begin{align*}
awx &= awa^{\otimes m,w} (w a)^m [(w a)^m]^\dagger \\
&= awa [(w a)^{\otimes m,w}]^2 (w a)^m [(w a)^m]^\dagger \\
&= a[(w a)^{\otimes m,w}]^2 [(w a)^m]^2 [(w a)^m]^\dagger \\
&= a^{\otimes m,w} (w a)^m [(w a)^m]^\dagger
\end{align*}
\]

Moreover, we have

\[
\begin{align*}
awx^2 &= awa^{\otimes m,w} (w a)^m [(w a)^m]^\dagger w a^{\otimes m,w} (w a)^m [(w a)^m]^\dagger \\
&= awa [(w a)^{\otimes m,w}]^2 (w a)^m [(w a)^m]^\dagger w a [(w a)^{\otimes m,w}]^2 (w a)^m [(w a)^m]^\dagger \\
&= awa [(w a)^{\otimes m,w}]^2 [(w a)^{\otimes m,w}]^2 [(w a)^m]^\dagger \\
&= a[(w a)^{\otimes m,w}]^2 [(w a)^m]^\dagger \\
&= a^{\otimes m,w} (w a)^m [(w a)^m]^\dagger \\
&= x.
\end{align*}
\]

(2) \( \Rightarrow \) (1) Suppose that the equation system

\[
awx = a(wa)^{\otimes m,p(wa)^m}, a(wx)^2 = x
\]
is consistent. In view of [1, Corollary 2.4], we have

\[ x = (awx)wx = (a(wa)\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger})wx \]
\[ = a(wa)\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger}wa(wx)^2 \]
\[ = a(wa)\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger}(wa)^m(wx)^{m+1} \]
\[ = a(wa)\overline{\otimes}_m(wa)^m(wx)^{m+1} \]
\[ = a(wa)\overline{\otimes}_mwx \]
\[ = a[(wa)\overline{\otimes}_{m-1}]^2w(awx) \]
\[ = a[(wa)\overline{\otimes}_{m-1}]^2w[a(wa)\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger}] \]
\[ = a[(wa)\overline{\otimes}_{m-1}]^2[wa][wa\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger}] \]
\[ = a[(wa)\overline{\otimes}_m]2[(wa)^m][(wa)^m]^{\dagger} \]
\[ = a\overline{\otimes}_m,(wa)^m[(wa)^m]^{\dagger} \]
\[ = a\overline{\otimes}_m, \]

as asserted. \(\square\)

Let \(a \in A^{\overline{\otimes}_m,}\). In view of Theorem 4.1, \(a^{\overline{\otimes}_m,} = a\overline{\otimes}_m(wa)^m[(wa)^m]^{\dagger}\). Set \(c = a(wa)\overline{\otimes}_m(wa)^m\). We now establish necessary and sufficient conditions under which \(a\) has weighted \(m\)-generalized core inverse.

**Theorem 4.3.** Let \(a \in A^{\overline{\otimes}_m,}\). The following are equivalent:

1. \(a^{\overline{\otimes}_m,} = x\).
2. \(awx = c[(wa)^m]^{\dagger}\) and \(xA \subseteq d,wa\).
3. \(awx = c[(wa)^m]^{\dagger}\) and \((a(wx))^2 = x\).

**Proof.** (1) \(\Rightarrow\) (2) In view of Theorem 4.1, we have

\[ awx = a(wa)^m[(wa)^m]^{\dagger} \]
\[ = c[(wa)^m]^{\dagger}. \]

By virtue of Theorem 2.1, we have

\[ xA = a\overline{\otimes}_m,(wa)^m[(wa)^m]^{\dagger}A \]
\[ \subseteq a\overline{\otimes}_m,waA \]
\[ = c[(wa)^m]^{\dagger}A \]
\[ \subseteq a(wa)^dA \]
\[ \subseteq a[(wa)^d]^{\dagger}A \]
\[ \subseteq a,wa. \]

(2) \(\Rightarrow\) (1) Since \(awx = c[(wa)^m]^{\dagger}\), we have \(awx = a(wa)^m[(wa)^m]^{\dagger} = a(wa)^m[(wa)^m]^{\dagger} = a\overline{\otimes}_m,(wa)^m[(wa)^m]^{\dagger}. \)
Since $xA \subseteq a^{d,w}A$, we derive that $a^{d,w}wawa^{d,w} = a^{d,w}$. Hence, $a^{d,w}wawx = x$. In view of Theorem 2.1, $a(wa)^{\oplus_m} \subseteq a[(wa)^{d}]^{2}wA = (aw)^{d}A$. Then $(aw)^{d}awa^{m} = a(wa)^{\oplus_{m}}$. We deduce that

$$x = a^{d,w}w(awx) = a[(wa)^{d}]^{2}w(awx)$$

$$= a[(wa)^{d}]^{2}wawa^{\oplus_{m},w}(wa)^{m}[(wa)^{m}]^{\dagger}$$

$$= (aw)^{d}awa^{\oplus_{m},w}(wa)^{m}[(wa)^{m}]^{\dagger}$$

$$= (aw)^{d}awa^{[m]}(wa)^{m}[(wa)^{m}]^{\dagger}$$

$$= a[(wa)^{\oplus_{m}}]^{2}(wa)^{m}[(wa)^{m}]^{\dagger}$$

$$= a^{\oplus_{m},w}(wa)^{m}[(wa)^{m}]^{\dagger}.$$ 

Therefore $a^{\oplus_{m},w} = x$, as desired.

(1) $\Rightarrow$ (3) By the argument above, we have $awx = c[(wa)^{m}]^{\dagger}$. In view of Theorem 2.1, $x = a^{\oplus_{m},w}(wa)^{m}[(wa)^{m}]^{\dagger}$. By using Corollary 4.2, we have $a(wx)^{2} = x$, as required.

(3) $\Rightarrow$ (1) Since $awx = c[(wa)^{m}]^{\dagger}$, we see that $awx = a(wa)^{\oplus_{m}}(wa)^{m}[(wa)^{m}]^{\dagger}$. As $a(wx)^{2} = x$, by virtue of Corollary 4.2, $x = a^{\oplus_{m},w}(wa)^{m}[(wa)^{m}]^{\dagger} = a^{\oplus_{m},w}$, as required. 

Let $X \in \mathbb{C}^{n \times n}$. The symbol $\mathcal{R}(X)$ denote the range space of $X$. We now derive

**Corollary 4.4.** Let $A \in \mathbb{C}^{n \times n}$. The following are equivalent:

1. $A^{W,\dagger} = X$.
2. $AWX = AA^{W}AA^{\dagger}$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^{P})$.
3. $AWX = A(WA)^{\oplus WAA^{\dagger}}$ and $A(WX)^{2} = X$.

**Proof.** Since $A \in \mathbb{C}^{n \times n}$, we easily see that $A^{\oplus_{1},w} = A^{W,\dagger}$. Therefore we complete the proof by Theorem 4.3. 

We are now ready to prove the following.

**Theorem 4.5.** Let $a, w \in A$. Then the following are equivalent:

1. $a^{\oplus_{m},w} = x$.
2. $xwcwx = x, awx = c[(wa)^{m}]^{\dagger}$ and $xwc = (aw)^{d}c$. 


Proof. (1) ⇒ (2) In view of Theorem 4.1, \( x = a^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger \). By virtue of Theorem 4.3, \( awx = c[(wa)^m]^\dagger \). Moreover, we verify that

\[
\begin{align*}
  xwcw &= a^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger[wa(\allowbreak wa)^{\otimes_m,w}(wa)^m]\allowbreak wa^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger \\
  &= a[(wa)^{\otimes_m,w}]^2\allowbreak[(wa)^m]^\dagger[wa(\allowbreak wa)^{\otimes_m,w}(wa)^m]\allowbreak wa^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger \\
  &= a[(wa)^{\otimes_m,w}]^2wa(\allowbreak wa)^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger \\
  &= a[(wa)^{\otimes_m,w}]^2(wa)^m[(wa)^m]^\dagger \\
  &= a^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger = x.
\end{align*}
\]

\[
xwc = a^{\otimes_m,w}(wa)^m[(wa)^m]^\dagger w[a(\allowbreak wa)^{\otimes_m,w}(wa)^m] \\
  = a[(wa)^{\otimes_m,w}]^2[(wa)^m]^\dagger wa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = a[(wa)^{\otimes_m,w}]^2wa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = a[(wa)^{\otimes_m,w}]^2(wa)^m \\
  = (aw)^dwa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = (aw)^d(wa)^dwa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = (aw)^dwa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = (aw)^d[(wa)^{\otimes_m,w}]^2(wa)^m \\
  = (aw)^d(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = (aw)^dwa(\allowbreak wa)^{\otimes_m,w}(wa)^m \\
  = (aw)^dwaCx.
\]

as required.

(2) ⇒ (1) By hypothesis, we check that

\[
\begin{align*}
x &= xwcw = (xwc)wx = [(aw)^dCx]wx \\
  &= a\allowbreak[(wa)^d]^2wx \\
  &\in a^{\otimes_m,w}_A.
\end{align*}
\]

According to Theorem 4.3, we complete the proof. □

**Corollary 4.6.** Let \( A \in \mathbb{C}^{n\times n} \) and \( C = A(WA)^{\otimes\allowbreak W}(WA) \). The following are equivalent:

1. \( A^{W\dagger} = X \).
2. \( XWCA = X, AWX = C(WA)^\dagger \) and \( XWC = (AW)^{D\dagger}C \).

**Proof.** It is immediate by Theorem 4.5 by choosing \( m = 1 \). □

5. APPLICATIONS

The purpose of this section is to give the applications of the \( w \)-weighted \( m \)-generalized group (core) inverse in solving the matrix equations. We consider the following equation in \( A \):

\[
[(wa)^d]^s(wa)^{m+1}wx = [(wa)^d]^s(wa)^mb \quad (5.1)
\]
where \(a, w, b \in A\) and \(m \in \mathbb{N}\).

**Theorem 5.1.** Let \(a \in A^{\oplus w}\). Then Eq. (5.1) has solution

\[
x = a^{\oplus w}b + [1 - a^{\oplus w}waw]y,
\]

where \(y \in A\) is arbitrary.

**Proof.** Let \(x = a^{\oplus w}b + [1 - a^{\oplus w}waw]y\), where \(y \in A\). Then

\[
w x = wa[(wa)^{\oplus m}]^2b + w[1 - a((wa)^{\oplus m})^2waw]y = (wa)^{\oplus m}b + [w - (wa)^{\oplus m}waw]y.
\]

Since \([(wa)^d]^*(wa)^{m+1}(wa)^{\oplus m} = (wa)^m\), we verify that

\[
[(wa)^d]^*[(wa)^{m+1}w.x = [(wa)^d]^*[wa[(wa)^{\oplus m}]^2b + w[1 - a((wa)^{\oplus m})^2waw]y = (wa)^d]^*[wa]^m b + [(wa)^d]^*[(wa)^{m+1}w - (wa)^d]^*[wa]^m waw]y = [(wa)^d]^*[wa]^m b,
\]

as asserted. \(\square\)

**Corollary 5.2.** Let \(a \in A^{\oplus w}\). Then the general solution of Eq. (5.1) is

\[
x = a^{\oplus w}b + [1 - a^{\oplus w}waw]y,
\]

where \(y \in A\) is arbitrary.

**Proof.** Let \(x\) be the solution of the Eq. (5.1). Then

\[
[(wa)^d]^*[(wa)^{m+1}w.x = [(wa)^d]^*[wa]^m b.
\]
In view of Theorem 3.1, \( a^{\hat{w}m.} = [a^{\hat{w}m.}]^{m+1} (aw)^{m-1}a \). Then
\[
a^{\hat{w}m.}wawx = [a^{\hat{w}m.}]^{m+1} (aw)^{m-1} awawx
\]
\[
= [a^{\hat{w}m.}]^{m+1} (aw)^{m-1} x
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2] (aw)^{m+1} x
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2][wa(\hat{w})^3^3] (wa)^{m+1} wx
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2][wa(\hat{w})^3^3](wa)^{m+1} wx
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2][wa(\hat{w})^3^3]*[wa(\hat{w})^3^3] (wa)^m b
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2][wa(\hat{w})^3^3]^2 w (aw)^m b
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2] (aw)^m b
\]
\[
= [a^{\hat{w}m.}]^m [a(\hat{w})^2^2] (aw)^m b
\]
\[
= a^{\hat{w}m.} b.
\]

Accordingly,
\[
x = a^{\hat{w}m.} b + [1 - a^{\hat{w}m.} waw] x.
\]

By using Theorem 5.1, we complete the proof.

\[\square\]

**Corollary 5.3.** Let \( a \in \mathcal{A}^{\hat{w}m.} \). If \( x \) is the solution of Eq. (5.1) and \( \text{im}(x) \subseteq \text{im}((aw)^d) \), then
\[
x = a^{\hat{w}m.} b.
\]

**Proof.** By virtue of Theorem 5.1, \( a^{\hat{w}m.} b \) is a solution of Eq. (5.1). Let \( x_1, x_2 \in \mathcal{A} \) be the solutions of Eq. (5.1) and satisfy \( \text{im}(x_i) \subseteq \text{im}((aw)^d) \). Write \( x_1 = (aw)^d y_1 \) and \( x_2 = (aw)^d y_2 \). Then \( x_1 - x_2 = [(aw)^d]^2 a(awaw)(x_1 - x_2) \). Hence, \( \text{im}(x_1 - x_2) \subseteq \text{im}((aw)^d) \). By hypothesis, we have
\[
[(wa)^d]^* (wa)^m b
\]
for \( i = 1, 2 \). Then \( [(wa)^d]^* (wa)^m b = 0 \); and so
\[
[(wa)^d]^* (wa)^m b
\]
By using Cline’s formula, we have \( w[(aw)^d]^2 a = (wa)^d \), and then
\[
[(wa)^d]^* (wa)^d (aw)^m b w(x_1 - x_2) = 0.
\]

Since the involution is proper, we have \( (aw)^d (wa)^m b w(x_1 - x_2) = 0 \); whence, \( (aw)(aw)^d (x_1 - x_2) = 0 \). Thus, \( x_1 = aw(aw)^d x_1 = aw(aw)^d x_2 = x_2 \). Therefore \( x = a^{\hat{w}m.} b \) is the unique solution of Eq. (5.1).

\[\square\]
Consider the following matrix equation:

\[
(WA)^D(WA)^{m+1}WX = [(WA)^D]^*(WA)^mB,
\]

where \( A \in \mathbb{C}^{q \times n}, W \in \mathbb{C}^{n \times q}, B \in \mathbb{C}^{n \times p} \) and \( m \in \mathbb{N} \).

**Corollary 5.4.** (1) The general solution of Eq. (5.2) is

\[
X = A^\mathbb{W}_mW^*B + [I_n - A^\mathbb{W}_mWAW]Y,
\]

where \( Y \in \mathbb{C}^{n \times p} \) is arbitrary.

(2) If \( X \) is the solution of Eq. (5.2) and \( \mathcal{R}(X) \subseteq \mathcal{R}((AW)^D) \), then

\[
X = A^\mathbb{W}_mW^*B.
\]

**Proof.** This is obvious by Corollary 5.2 and Corollary 5.3. \( \square \)

Let \( a \in \mathbb{A}^{\mathbb{G}_m,w} \). We now come to consider the following equation in \( \mathbb{A} \):

\[
[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*[(wa)^m]^*b,
\]

where \( a, w, b \in \mathbb{A} \) and \( m \in \mathbb{N} \). The following lemma is crucial.

**Lemma 5.5.** Let \( a \in \mathbb{A}^{\mathbb{G}_m,w} \). Then \( a \in \mathbb{A}^{\mathbb{G}_m,w} \).

**Proof.** By hypothesis, \( a \in \mathbb{A}^{\mathbb{G}_m,w} \cap \mathbb{A}^{\mathbb{G}_m} \). In light of [1, Theorem 2.1], \((wa)^m \in \mathbb{A}^{\mathbb{G}_m} \cap \mathbb{A}^{\mathbb{G}_m} \). By virtue of [4, Theorem 3.1], \((wa)^m \in \mathbb{A}^{\mathbb{G}_m} \). Evidently, \((wa)^m = (wa)^{m-1}[(wa)^m]^{\mathbb{G}_m} \). Accordingly, \( a \in \mathbb{A}^{\mathbb{G}_m,w} \) by [3, Theorem 2.1]. \( \square \)

We are ready to prove:

**Theorem 5.6.** Let \( a \in \mathbb{A}^{\mathbb{G}_m,w} \). Then the general solution of Eq. (5.3) is

\[
x = a^{\mathbb{G}_m,w}b + [1 - a^{\mathbb{G}_m,w}waw]y,
\]

where \( y \in \mathbb{A} \) is arbitrary.

**Proof.** Let \( x = a^{\mathbb{G}_m,w}b + [1 - a^{\mathbb{G}_m,w}waw]y \), where \( y \in \mathbb{A} \). In view of Theorem 4.1, \( a^{\mathbb{G}_m,w} = a^{\mathbb{G}_m,w}(wa)^m[(wa)^m]^* \). Then

\[
x = a^{\mathbb{G}_m,w}(wa)^m[(wa)^m]^*b + [1 - a^{\mathbb{G}_m,w}waw]y.
\]

By virtue of Theorem 5.1, \( x \) is the solution of Eq. (5.3).

In light of Lemma 5.5, \( a \in \mathbb{A}^{\mathbb{G}_m,w} \). By using Corollary 5.2,

\[
x = a^{\mathbb{G}_m,w}(wa)^m[(wa)^m]^*b + [1 - a^{\mathbb{G}_m,w}waw]y
\]

is the general solution of Eq. (5.3), as required. \( \square \)
Corollary 5.7. Let $a \in \mathcal{A}^{\oplus m,w}$. If $x$ is the solution of Eq. (5.3) and $\text{im}(x) \subseteq \text{im}((aw)^d)$, then
\[ x = a^{\oplus m,w}b. \]

Proof. By virtue of Theorem 5.6, $a^{\oplus m,w}b$ is a solution of Eq. (5.3). Let $x_1, x_2 \in \mathcal{A}$ be the solutions of Eq. (5.3) and satisfy $\text{im}(x_i) \subseteq \text{im}((aw)^d)$. Then they are solutions of the equation:
\[ [(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^m[(wa)^m]^\dagger b, \]
as desired. \hfill \Box

Consider the following matrix equation:
\[
[(WA)^D]^*(WA)^{m+1}WX = [(WA)^D]^*(WA)^2m[(WA)^m]^\dagger B, \qquad (5.4)
\]
where $A \in \mathbb{C}^{q \times n}, W \in \mathbb{C}^{n \times q}, B \in \mathbb{C}^{n \times p}$ and $m \in \mathbb{N}$.

Corollary 5.8. (1) The general solution of Eq. (5.4) is
\[
X = A^{\oplus m,w}B + [I_n - A^{\oplus m,w}WAW]Y,
\]
where $Y \in \mathbb{C}^{n \times p}$ is arbitrary.

(2) If $X$ is the solution of Eq. (5.4) and $\mathcal{R}(X) \subseteq \mathcal{R}((AW)^D)$, then
\[
X = A^{\oplus m,w}B.
\]

Proof. This is obvious by Theorem 5.5 and Corollary 5.6. \hfill \Box

References

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