An integral collocation method

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A new method is developed of which is applied to a problem involving a 1D wave equation in disguise.

1. Problem description

Let $u = u(x, y) \in \mathbb{R}$ with $x, y \in \mathbb{R}$. A 1D wave equation in disguise is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

where we take $c = 1$. The boundary conditions are

$$u = 0 \text{ at } x = -1 \text{ for } y > -1, \tag{2}$$

$$u = 0 \text{ at } x = 1 \text{ for } y > -1, \tag{3}$$

$$u - \cos(\frac{\pi}{2}x) = 0 \text{ at } y = -1 \text{ for } -1 < x < 1, \tag{4}$$

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = -1 \text{ for } -1 < x < 1. \tag{5}$$

On replacing the spatial variable y with the (time-delayed) variable $\tau = t - 1$ where t is the time variable, above can be recognised as the wave equation in one space dimension. Then $u(x, t)$ denotes the transverse displacement of a tightly-stretched vibrating string. The string is fastened on the x axis at $x = \pm 1$ so that $u(\pm 1, t) = 0$. Here we have $u(x, 0) = \cos(\frac{\pi}{2}x)$ is the initial shape of the string. The string is released from rest so that the initial transverse velocity distribution $u_t(x, 0) = 0$. Here the constant c is given by $c^2 = T/\rho$ where T is the tension in the string and ρ is the density (mass/unit length) of the string. It is known that here c is the speed at which the transverse waves propagate along the string.

2. Exact solution

We can find the exact solution to the problem in §1 for $x \in [-1, 1]$ and $\forall\ y \geqslant -1$. By the method of separation of variables assume a solution of the form

$$u = XY \tag{6}$$

where $X = X(x)$ and $Y = Y(y)$. Then (1) implies

$$X''Y - XY'' = 0 \tag{7}$$

which implies

$$X'' = -\lambda X \tag{8}$$

and

$$Y'' = -\lambda Y \tag{9}$$

where λ is the separation constant. The boundary conditions (2), (3) imply

$$X(-1) = X(1) = 0. \tag{10}$$

For nontrivial solutions we have $\lambda = p^2 > 0$, then (8) implies

$$X = c_0 \cos(px) + c_1 \sin(px) \tag{11}$$

where c_0, c_1 are arbitrary constants. The boundary conditions (10) then imply

$$c_0 \cos(p) - c_1 \sin(p) = 0 \tag{12}$$

and

$$c_0 \cos(p) + c_1 \sin(p) = 0. \tag{13}$$

Therefore we have case I : $p = n\pi$, $X_n = c_{1,n}\sin(n\pi x)$, $\lambda_n = (n\pi)^2, n = 1, 2, \ldots, \infty$ or case II : $p = \frac{\pi}{2} + n\pi$, $X_n = c_{0,n}\cos((\frac{\pi}{2} + n\pi)x)$, $\lambda_n = (\frac{\pi}{2} + n\pi)^2, n = 0, 1, 2, \ldots, \infty$. Here $c_{0,n}, c_{1,n}$ are arbitrary constants. For case I we can not match the boundary condition (4). For case II we have that (9) becomes

$$Y_n'' = -(\frac{\pi}{2} + n\pi)^2 Y_n \tag{14}$$

which implies

$$Y_n = c_{2,n}\cos((\frac{\pi}{2} + n\pi)y) + c_{3,n}\sin((\frac{\pi}{2} + n\pi)y) \tag{15}$$

where $c_{2,n}, c_{3,n}$ are arbitrary constants. Using the superposition principle we then have

$$u = \sum_{n=0}^{\infty} c_{0,n}\cos((\frac{\pi}{2} + n\pi)x)[c_{2,n}\cos((\frac{\pi}{2} + n\pi)y) + c_{3,n}\sin((\frac{\pi}{2} + n\pi)y)]. \tag{16}$$

Then

$$\frac{\partial u}{\partial y} = \sum_{n=0}^{\infty} c_{0,n}\cos((\frac{\pi}{2} + n\pi)x)[-c_{2,n}\sin((\frac{\pi}{2} + n\pi)y) + c_{3,n}\cos((\frac{\pi}{2} + n\pi)y)][\frac{\pi}{2} + n\pi]. \tag{17}$$

The boundary condition (4) implies

$$\sum_{n=0}^{\infty} c_{0,n}\cos((\frac{\pi}{2} + n\pi)x)[c_{2,n}\cos((\frac{\pi}{2} + n\pi)) - c_{3,n}\sin((\frac{\pi}{2} + n\pi))] = \cos(\frac{\pi}{2}x). \tag{18}$$

which implies

$$c_{0,0}[c_{2,0}\cos(\frac{\pi}{2}) - c_{3,0}\sin(\frac{\pi}{2})] = 1. \tag{19}$$

and

$$c_{0,n}[c_{2,n}\cos(\frac{\pi}{2} + n\pi) - c_{3,n}\sin(\frac{\pi}{2} + n\pi)] = 0 \tag{20}$$

for $n > 0$. The boundary condition (5) implies

$$\sum_{n=0}^{\infty} c_{0,n}\cos((\frac{\pi}{2} + n\pi)x)[c_{2,n}\sin(\frac{\pi}{2} + n\pi) + c_{3,n}\cos(\frac{\pi}{2} + n\pi)][\frac{\pi}{2} + n\pi] = 0 \tag{21}$$

which implies

$$c_{0,n}[c_{2,n}\sin(\frac{\pi}{2} + n\pi) + c_{3,n}\cos(\frac{\pi}{2} + n\pi)] = 0 \tag{22}$$

for $n \geqslant 0$. Therefore we have

$$c_{0,0}[c_{2,0}\cos(\frac{\pi}{2}) - c_{3,0}\sin(\frac{\pi}{2})] = 1, \tag{23}$$

$$c_{0,0}[c_{2,0}\sin(\frac{\pi}{2}) + c_{3,0}\cos(\frac{\pi}{2})] = 0 \tag{24}$$

and $c_{0,n} = 0$ for $n > 0$. This implies $c_{0,0} = 1, c_{2,0} = 0, c_{3,0} = -1$. Our final exact solution is then

$$u = -\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) \tag{25}$$

of which can be checked by substitution into the problem in §1.

3. An integral collocation method

This method was inspired by $[1, 2, 3]$. We solve the problem in §1 numerically on $x \in [-1, 1], y \in [-1, 1]$ as follows. We start with the expansion

$$\frac{\partial^2 u}{\partial x^2} = \sum_{l=1}^{N+1} \tilde{u}_l(y) T_{l-1}(x) \tag{26}$$

where $T_l(x)$ are Chebyshev functions with $\tilde{u}_l(y)$ as the unknown coefficients, and N is a positive integer. We define the vector

$$\bar{\mathbf{u}}^{(n)} = [\frac{\partial^n u}{\partial x^n}|_{x=x_1}, \frac{\partial^n u}{\partial x^n}|_{x=x_2}, \dots, \frac{\partial^n u}{\partial x^n}|_{x=x_{N+1}}]^T \tag{27}$$

for $n = 0, 1, 2$. Here x_i are the Gauss–Lobatto points

$$x_i = \cos(\pi((i-1)/N)) \ \text{ for } \ i = 1, 2, \dots, N+1. \tag{28}$$

Then we have

$$\bar{\mathbf{u}}^{(1)} = \widehat{W}\bar{\mathbf{u}}^{(2)} + \frac{\partial u}{\partial x}|_{x=-1}\mathbf{1} \tag{29}$$

where $\widehat{W}$ is the integration matrix [2] for integrating with respect to x and $\mathbf{1}$ is a vector with all entries equal to one. It then follows that

$$\bar{\mathbf{u}}^{(0)} = \widehat{W}^2\bar{\mathbf{u}}^{(2)} + \widehat{W}\frac{\partial u}{\partial x}|_{x=-1}\mathbf{1} + u|_{x=-1}\mathbf{1}. \tag{30}$$

The partial differential equation (1) at $x = x_j$ is

$$\sum_{l=1}^{N+1} \tilde{u}_l(y)T_{l-1}(x_j) - \sum_{q=1}^{N+1}(\widehat{W}^2)_{j,q}\sum_{l=1}^{N+1}\frac{\partial^2\tilde{u}_l(y)}{\partial y^2}T_{l-1}(x_q) - \frac{\partial^2}{\partial y^2}(\frac{\partial u}{\partial x}|_{x=-1})\sum_{q=1}^{N+1}\widehat{W}_{j,q} = 0. \tag{31}$$

The boundary condition (2) implies

$$u|_{x=-1} = 0. \tag{32}$$

The boundary condition (3) implies

$$\sum_{q=1}^{N+1}(\widehat{W}^2)_{1,q}\sum_{l=1}^{N+1}\tilde{u}_l(y)T_{l-1}(x_q) + (\frac{\partial u}{\partial x}|_{x=-1})\sum_{q=1}^{N+1}\widehat{W}_{1,q} = 0 \tag{33}$$

which implies

$$\frac{\partial u}{\partial x}|_{x=-1} = \frac{-\sum_{q=1}^{N+1}(\widehat{W}^2)_{1,q}\sum_{l=1}^{N+1}\tilde{u}_l(y)T_{l-1}(x_q)}{\sum_{L=1}^{N+1}\widehat{W}_{1,L}}. \tag{34}$$

The partial differential equation (1) at $x = x_j$ becomes

$$\sum_{l=1}^{N+1}\tilde{u}_l(y)T_{l-1}(x_j) - \sum_{q=1}^{N+1}(\widehat{W}^2)_{j,q}\sum_{l=1}^{N+1}\frac{\partial^2\tilde{u}_l(y)}{\partial y^2}T_{l-1}(x_q) + \frac{\sum_{q=1}^{N+1}(\widehat{W}^2)_{1,q}\sum_{l=1}^{N+1}\frac{\partial^2\tilde{u}_l(y)}{\partial y^2}T_{l-1}(x_q)}{\sum_{L=1}^{N+1}\widehat{W}_{1,L}}\sum_{Q=1}^{N+1}\widehat{W}_{j,Q} = 0. \tag{35}$$

Now let

$$\frac{\partial^2\tilde{u}_l(y)}{\partial y^2} = \sum_{m=1}^{M+1}\hat{u}_{l,m}T_{m-1}(y) \tag{36}$$

where $\hat{u}_{l,m}$ are unknown coefficients to be found and M is a positive integer. We define the vector

$$\tilde{\mathbf{u}}_{\mathbf{l}}(\mathbf{y})^{(n)} = [\frac{\partial^n\tilde{u}_l(y)}{\partial y^n}|_{y=y_1}, \frac{\partial^n\tilde{u}_l(y)}{\partial y^n}|_{y=y_2}, \dots, \frac{\partial^n\tilde{u}_l(y)}{\partial y^n}|_{y=y_{M+1}}]^T \tag{37}$$

for $n = 0, 1, 2$. Here y_i are the Gauss–Lobatto points

$$y_i = \cos(\pi((i-1)/M)) \ \text{ for } \ i = 1, 2, \dots, M+1. \tag{38}$$

Then we have

$$\tilde{\mathbf{u}}_{\mathbf{l}}(\mathbf{y})^{(1)} = \widehat{\widehat{W}}\tilde{\mathbf{u}}_{\mathbf{l}}(\mathbf{y})^{(2)} + \frac{\partial\tilde{u}_l(y)}{\partial y}|_{y=-1}\tilde{\mathbf{1}} \tag{39}$$

where $\widehat{\widehat{W}}$ is the integration matrix [2] for integrating with respect to y and $\tilde{\mathbf{1}}$ is a vector with all entries equal to one. It then follows that

$$\tilde{\mathbf{u}}_{\mathbf{l}}(\mathbf{y})^{(0)} = \widehat{\widehat{W}}^2\tilde{\mathbf{u}}_{\mathbf{l}}(\mathbf{y})^{(2)} + \widehat{\widehat{W}}\frac{\partial\tilde{u}_l(y)}{\partial y}|_{y=-1}\tilde{\mathbf{1}} + \tilde{u}_l(-1)\tilde{\mathbf{1}}. \tag{40}$$

The partial differential equation (1) at $x = x_j$, $y = y_k$ is

$$\sum_{l=1}^{N+1}[\sum_{s=1}^{M+1}(\widehat{\widehat{W}}^2)_{k,s}\sum_{m=1}^{M+1}\hat{u}_{l,m}T_{m-1}(y_s)+\sum_{s=1}^{M+1}\widehat{\widehat{W}}_{k,s}\frac{\partial \tilde{u}_l(y)}{\partial y}|_{y=-1}+\tilde{u}_l(-1)]T_{l-1}(x_j)$$
$$-\sum_{q=1}^{N+1}(\widehat{W}^2)_{j,q}\sum_{l=1}^{N+1}\sum_{m=1}^{M+1}\hat{u}_{l,m}T_{m-1}(y_k)T_{l-1}(x_q)+\frac{\sum_{q=1}^{N+1}(\widehat{W}^2)_{1,q}\sum_{l=1}^{N+1}\sum_{m=1}^{M+1}\hat{u}_{l,m}T_{m-1}(y_k)T_{l-1}(x_q)}{\sum_{L=1}^{N+1}\widehat{W}_{1,L}}\sum_{Q=1}^{N+1}\widehat{W}_{j,Q}=0. \quad (41)$$

Next we note from our previous expansions defined above that it can be deduced easily that

$$u=\sum_{l=1}^{N+1}[\sum_{m=1}^{M+1}\hat{u}_{l,m}\int_{-1}^{y}\int_{-1}^{\hat{\hat{y}}}T_{m-1}(\hat{y})\,d\hat{y}\,d\hat{\hat{y}}+\frac{\partial \tilde{u}_l(y)}{\partial y}|_{y=-1}(y+1)+\tilde{u}_l(-1)]\int_{-1}^{x}\int_{-1}^{\hat{\hat{x}}}T_{l-1}(\hat{x})\,d\hat{x}\,d\hat{\hat{x}}$$
$$-\frac{\sum_{q=1}^{N+1}(\widehat{W}^2)_{1,q}\sum_{l=1}^{N+1}[\sum_{m=1}^{M+1}\hat{u}_{l,m}\int_{-1}^{y}\int_{-1}^{\hat{\hat{y}}}T_{m-1}(\hat{y})\,d\hat{y}\,d\hat{\hat{y}}+\frac{\partial \tilde{u}_l(y)}{\partial y}|_{y=-1}(y+1)+\tilde{u}_l(-1)]T_{l-1}(x_q)(x+1)}{\sum_{L=1}^{N+1}\widehat{W}_{1,L}}. \quad (42)$$

At this point we use Maple to solve (41) and the remaining boundary conditions (4,5) for $\hat{u}_{l,m}$, $\frac{\partial \tilde{u}_l(y)}{\partial y}|_{y=-1}$, and $\tilde{u}_l(-1)$ where $l = 1, 2, \ldots, N+1$ and $m = 1, 2, \ldots, M+1$. Note the boundary conditions (4,5) are to be evaluated at a set of x points different to the x_i points in order for the number of equations to be solved to be equal to the number of variables it is to be solved for. For the boundary conditions (4,5), we use (42) with the different x points

$$\xi_i = \cos(\pi i/(N+2)) \text{ for } i = 1, 2, \ldots, N+1. \quad (43)$$

The Maple code is omitted. For $M = N = 20$, the obtained numerical solution is indistinguishable to the exact solution as in Figure 1 (left). The residual r is defined here as the outcome of substituting the numerical solution into the left hand side of the problem described in §1. For $M = N = 20$ we have $\max|r| \approx 0.015$. The error e is defined here as the difference between the exact solution and numerical solution. In Figure 1 (right) we plot e and see that this method is accurate at $M = N = 20$ with $\max|e| \approx 0.000015$.

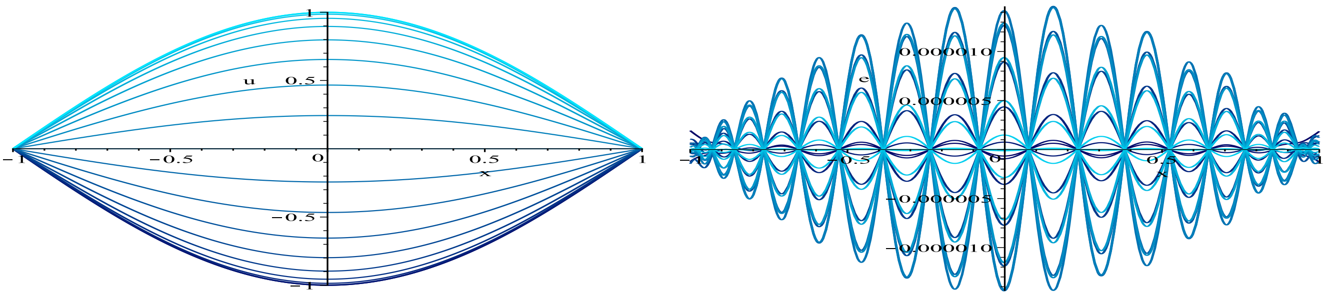

Figure 1: (left) Plot of the exact/numerical solution u vs x and (right) plot of the error e vs x at the Gauss–Lobatto points $y = y_i = \cos(\pi((i-1)/M))$ for $i = 1, 2, \ldots, M+1$. Here $M = N = 20$. The curves get darker as y increases from negative one.

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References

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