Clarifying an Early Step in Hardy's Transcendence of π Proof

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Abstract

We clarify and strengthen Hardy's footnote proof of an essential step in his proof of the transcendence of π. We show that ri is algebraic if and only if r is algebraic.

Introduction

On page 223 Hardy gives a proof that π is transcendental [1]. His proof shows that πi does not solve a integer polynomial, but technically this isn't showing π doesn't so solve an integer polynomial. He needs to show that the one implies the other. Here is his one line proof.

If $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ and $y = xi$ then

$$a_0y^n - a_2y^{n-2} + \cdots + i(a_1y^{n-1} - a_3y^{n-3} + \dots) = 0$$

and so

$$(a_0y^n - a_2y^{n-2} + \dots)^2 + (a_1y^{n-1} - a_3y^{n-3} + \dots)^2 = 0.$$

This is very condensed and presupposes that $n \equiv 0$ mod(4) which he doesn't stipulate. As just about all proofs of π's transcendence require this step, we wish to remove this potential stumbling block.

It should be noted that he is going to use a contrapositive argument in the main body of his proof: $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim q$. He needs to show that if r is algebraic then ri is algebraic. The contrapositive is *if ri is not algebraic then r is not algebraic*. His proof shows the antecedent true.

The Idea

The idea is easily demonstrated. Consider $f(x) = a_0x^4+a_1x^3+a_2x^2+a_3x^1+a_4x^0$ and suppose $f(r) = 0$. We can find a new set of coefficients of the same ilk as a_i such that if $g(x)$ has this set and $g(ri) = 0$. This can be done as $i^k \in \{i^0, i^1, i^2, i^3\} = \{1, i, -1, -i\}$. These powers of i correspond to classes from modulo 4 (remainders on division by 4) and any natural number power (our exponents) is in one of these classes.

Let the terms of an nth degree integer polynomial be given by $t_k(x) = a_{n-k}x^k$, then any $t_k(ri)$ reduces to one of $\{1, i, -1, -i\}$ times $t_k(r)$. Here's the details for our polynomial $f(x)$:

$$t_4(ri) = a_0(ri)^4 = a_0r^4i^4 = a_0r^4 = 1 \cdot t_4(r),$$
$$t_3(ri) = a_1(ri)^3 = a_1r^3i^3 = -ia_1r^3 = -i \cdot t_3(r),$$
$$t_2(ri) = a_2(ri)^2 = a_2r^2i^2 = -1a_2r^2 = -1 \cdot t_2(r),$$
$$t_1(ri) = a_3(ri)^1 = a_3r^1i^1 = ia_3r^1 = i \cdot t_1(r),$$
$$t_0(ri) = a_4(ri)^0 = a_4r^0i^0 = a_4r^0 = 1 \cdot t_0(r),$$

That's step one. Step two is to note that the right hand sides of the above equations when multiplied again by one of $\{1, i, -1, -i\}$ and then added re-constitute $f(r)$. The details:

$$t_4(ri) = 1 \cdot t_4(r) \text{ multiple by 1 for } t_4(r),$$
$$t_3(ri) = -i \cdot t_3(r) \text{ multiple by i for } t_3(r),$$
$$t_2(ri) = -1 \cdot t_2(r) \text{ multiple by } -1 \text{ for } t_2(r),$$
$$t_1(ri) = i \cdot t_1(r) \text{ multiple by -i for } t_1(r),$$
$$t_0(ri) = 1 \cdot t_0(r) \text{ multiple by 1 for } t_0(r),$$

Doing the directive in these equations and adding, we arrive at

$$t_4(ri) + it_3(ri) - t_2(ri) - it_1(ri) + t_0(ri) = t_4(r) + t_3(r) + t_2(r) + t_1(r) + t_0(r),$$

but this is 0. Letting $E(x)$ be alternating in sign even terms and $O(x)$ alternating odd terms, we note $E(ri) + iO(ri) = 0$. Consider $E(x)$ and $O(x)$ are free of imaginary coefficients, so we almost could make $g(x) = E(x) + iO(x)$ except for that i multiple. But $g(x) = |E(x) + iO(x)| = E^2(x) + O^2(x)$ has coefficients free of i in its coefficients. It works $g(ri) = 0$ with $g(x)$ having integer coefficients.

Looking back at Hardy's proof(?), you see what he is up to and also how he really does have to assume his n is divisible by 4. Can we tighten the idea up to a real proof without this assumption. Next.

The Proof

All functions are integer polynomials.

Theorem 1. *A number ri is an algebraic number if and only if r is an algebraic number.*

Proof. $\Rightarrow$ If ri is algebraic there exists functions such that $f(ir) = E(r) + iO(r) = 0$ with $E(r)$ and $O(r)$ integer polynomials without any terms with i in them. It follows that $E(r) = O(r) = 0$. That is r is algebraic.

$\Leftarrow$ If r is algebraic, there exists a function $f(r) = 0$. Each term of this function will be of the form $T_j(x) = a_j x^{n-j}$. The degree of each term will be in one of the four modulo 4 classes: $[0], [1], [2]$ or $[3]$. Each term will have a multiple $m \in \{1, i, -1, -i\}$ such that $T_j(xi) = mT_j(x)$. Using these terms form $New(ri) = E(ri) + iO(ri) = f(r) = 0$ where E are alternating even terms and O are alternating odd terms. As $|New(x)| = E^2(x) + O^2(x) = 0$ is a integer polynomial such that it is 0 and ri, we have shown ri is algebraic. $\square$

Conclusion

There are places in Hardy's classic where he has an untoward step like this one. He leaves a lot to the reader. If the reader is steeped in techniques and can accept his word that a laborsome proof can be given, then all is well. But a novice reader might become forlorn at such fair. I hope this article helps such.

References

[1] G.H. Hardy and E.M.Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford 2008.