# A new continued fraction approximation for the Lugo and Euler–Mascheroni constants

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**ABSTRACT:** In this paper, we provide a remarkable method for construction of continued fraction based on a given power series. Then we establish a new continued fraction approximation for the Lugo and Euler–Mascheroni constants. Especially, we analytically determine the coefficients of the Lugo's asymptotic formula and all parameters of the continued fraction by Bernoulli numbers.

**Keywords**: Lugo's constant, Euler–Mascheroni constant, continued fraction, psi function

## 1. Introduction

The mathematical constants and special functions such as the Euler-Mascheroni constant and the gamma function arise in many fields of pure and applied mathematics such as theory of probability, applied statistics, number theory and so on.

The Euler-Mascheroni constant $\gamma$, now universally known as gamma, was introduced by the Swiss mathematician Leonhard Euler (1707-1783) in 1734, which is defined as the limit of the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n. \tag{1.1}$$

The constant $\gamma$ is generally accepted to be the most significant of the 'constant' and as such is the important special constant of mathematics, after $\pi$ and e. It is deeply related to the gamma function $\Gamma(z)$ by means of the familiar Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [12, Chapter 1, Section 1.1]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[ \left(1 + \frac{z}{n}\right) e^{-z/n} \right] \quad \left( |z| < \infty \right). \tag{1.2}$$

Lugo[3] considered the sequence $L_n$, which is essentially an interesting analogue of the sequence $D_n$, defined by

$$L_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j} - (2\ln 2)n + \ln n . \tag{1.3}$$

We can easily find that

$$L = \lim_{n\to\infty} L_n = -\frac{1}{2} - \gamma + \ln 2 \tag{1.4}$$

where L is called Lugo's constant.

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As you can see, the Lugo and Euler-Mascheroni constants are related to each other.

In this field, the most important problem is to find more accurate approximation for them, so during the past several decades, many mathematicians and scientists have worked on this subject.

Up to now, many researchers made great efforts in this area of establishing more accurate approximations for the Lugo and Euler-Mascheroni constant and had lots of inspiring results.

Lugo[9] proved the following asymptotic formula:

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2-\frac{5}{8n}+\frac{7}{48n^2}+o\left(n^{-3}\right) \ (n\to\infty). \tag{1.5}$$

Chen and Srivastava[4] established new analytical representations for the Euler Mascheroni constant $\gamma$:

$$\gamma=-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j}+\ln 2-1+\left(n+\frac{1}{2}\right)\psi\left(n+\frac{1}{2}\right)-\left(n+\frac{3}{2}\right)\psi(n)+(2\ln 2)n-\frac{3}{2n}\,(n\in N), \tag{1.6}$$

in terms of the psi (or digamma) function defined by $\psi(z)$ defined by

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)} \text{ or } \ln\Gamma(z)=\int_1^z\psi(t)dt. \tag{1.7}$$

Recently, authors have focused on continued fractions in order to obtain new approximations. [10,11,13-15]

For example, Lu[6] provided faster sequence convergent to $\gamma$ as follows.

$$r_{n,s}=H_n-\ln n-\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}, \tag{1.8}$$

where

$$\mathrm{H}_n=\sum_{k=1}^{n}\frac{1}{k}$$

is the $n^{\text{th}}$ harmonic number and

$$a_1=\frac{1}{2},a_2=\frac{1}{6},a_4=\frac{3}{5},a_6=\frac{79}{126},a_8=\frac{7230}{6241},a_{10}=\frac{4146631}{3833346},\cdots.$$
$$a_{2k+1}=-a_{2k}\ (1\le k\le 6)$$

Moreover, he used continued fraction approximation to consider new classes of sequences for the Euler–Mascheroni constant as follows.[7]

$$L_{r,n}=H_{n-1}+\frac{1}{rn}-\ln n-\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}\ (r\neq 2),$$
$$L_{2,n}=H_{n-1}+\frac{1}{2n}-\ln n-\frac{1}{n}\cfrac{b_1}{n+\cfrac{b_2}{n+\cfrac{b_3}{n+\cfrac{b_4}{n+\ddots}}}} \tag{1.9}$$

where

$$a_1 = \frac{2-r}{2r}, a_2 = \frac{r}{6(2-r)}, a_3 = \frac{r}{6(r-2)}, a_4 = \frac{3(2-r)}{5r}, \cdots$$
$$b_1 = -\frac{1}{12}, b_2 = \frac{1}{10}, b_3 = \frac{79}{210}, b_4 = \frac{1205}{1659}, \cdots$$

In [8], he introduced new classes of sequences.

$$L_{n,k} = H_n - \ln n - \frac{1}{k}\ln\left(1 + \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}\right), \tag{1.10}$$

where

$$a_1 = \frac{k}{2}, a_2 = \frac{2-3k}{12}, a_3 = \frac{3k^2+4}{12(3k-2)}, a_4 = -\frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{20(3k-2)(3k^2+4)}, \cdots$$

In [2], a sequence concerning the Lugo's constant is provided.

$$v_n = L - L_n - \cfrac{\frac{5}{8}}{n + a + \cfrac{b}{n + c + \cfrac{p}{n+q}}} \tag{1.11}$$

where.

$$a = \frac{7}{30}, b = \frac{53}{1800}, c = \frac{1339}{1590}, p = \frac{15975}{22472}, q = -\frac{6528287}{59267250}.$$

In this paper, we provide a method for construction of continued fraction based on a given power series. Then we establish a new continued fraction approximation for the Lugo and Euler–Mascheroni constants. Especially, we analytically determine the coefficients of the Lugo's asymptotic formula and all parameters of the continued fraction by Bernoulli numbers.

The rest of this paper is arranged as follows.

In Sect. 2, some useful lemmas are given. In Sect. 3, a main method to construct continued fraction and a new continued fraction approximation for the Lugo and Euler–Mascheroni constants are provided. In the last section, the conclusions are given.

## 2. Lemmas

In this section, some useful lemmas are given. Especially, we analytically determine the coefficients of the Lugo's asymptotic formula.

**Lemma 2.1.** The psi function $\psi$ has the asymptotic formulas as follows;

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2ix^{2i}} \quad , \quad x \to \infty \tag{2.1}$$

$$\psi\left(x+\frac{1}{2}\right)\sim \ln x-\sum_{i=1}^{\infty}\frac{B_{2i}(1/2)}{2ix^{2i}}=\ln x+\sum_{i=1}^{\infty}\frac{B_{2i}(1-2^{1-2i})}{2ix^{2i}}\quad,\qquad x\to\infty \tag{2.2}$$

where $B_n$ $(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$ denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\ |z|<2\pi, \tag{2.3}$$

then the first few terms of $B_n$ are as follows.

$$B_{2n+1}=0, n\geq 1,$$
$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{10}=\frac{5}{66},\cdots.$$

We can find expressions above by differentiating expressions (3.14) and (5.4) in [3].

**Lemma 2.2.** The following asymptotic formula holds true:

$$L_n\sim -\frac{1}{2}-\gamma+\ln 2+\sum_{i=1}^{\infty}\frac{c_i}{n^i}\ (n\to\infty), \tag{2.4}$$

where

$$c_1=-\frac{5}{8},$$
$$c_{2i}=\frac{(2-2^{-2i})B_{2i}}{2i}, c_{2i+1}=\frac{(2-2^{-1-2i})B_{2(i+1)}}{2(i+1)}_i,\quad i=1,2,\cdots.$$

**Proof.** From (1.3) and (1.6), we can get

$$L_n=-\frac{1}{2}-\gamma+\ln 2+\left(n+\frac{1}{2}\right)\psi\left(n+\frac{1}{2}\right)-\left(n+\frac{3}{2}\right)\psi(n)+(2\ln 2)n-\frac{3}{2n}-\frac{1}{2}+\ln n,\ n\in\mathbb{N}. \tag{2.5}$$

Substituting (2.1) and (2.2) into (2.5) ,

$$\begin{aligned}L_n\sim -\frac{1}{2}-\gamma+\ln 2+\left(n+\frac{1}{2}\right)&\left(\ln n+\sum_{i=1}^{\infty}\frac{B_{2i}(1-2^{-1-2i})}{2i\cdot n^{2i}}\right)\\ &-\left(n+\frac{3}{2}\right)\left(\ln n-\frac{1}{2n}-\sum_{i=1}^{\infty}\frac{B_{2i}}{2i\cdot n^{2i}}\right)-\frac{3}{2n}-\frac{1}{2}+\ln n\end{aligned}. \tag{2.6}$$

Thus,

$$L_n\sim -\frac{1}{2}-\gamma+\ln 2-\frac{3}{4n}+\sum_{i=1}^{\infty}\frac{B_{2i}(2-2^{1-2i})}{2i\cdot n^{2i-1}}+\sum_{i=1}^{\infty}\frac{B_{2i}(2-2^{-2i})}{2i\cdot n^{2i}}=-\frac{1}{2}-\gamma+\ln 2+\sum_{i=1}^{\infty}\frac{c_i}{n^i}, \tag{2.7}$$

where

$$c_1=-\frac{5}{8},$$
$$c_{2i}=\frac{(2-2^{-2i})B_{2i}}{2i}, c_{2i+1}=\frac{(2-2^{-1-2i})B_{2(i+1)}}{2(i+1)}_i,\quad i=1,2,\cdots.$$

The proof of Lemma 2.2 is complete.

**Remark 2.1.** As you can see, (2.4) is equivalent to Lugo's asymptotic formula (1.5). A remarkable point is that we analytically determine the coefficients of the asymptotic formula.

The first few terms of coefficients are as follows.

$$c_1=-\frac{5}{8}, c_2=\frac{7}{48}, c_3=-\frac{1}{64}, c_4=-\frac{31}{1920}, c_5=\frac{1}{128}, \cdots.$$

**Lemma 2.3.**(The Euler connection [5, p.19, Eq. (1.7.1, 1.7.2)]) Let $\{c_k\}$ be a sequence in $\mathbb{C}\setminus\{0\}$ and

$$f_n=\sum_{k=0}^{n} c_k , \quad \mathrm{n}\in\mathbb{N}_0. \tag{2.8}$$

Since $f_0\neq\infty$, $f_n\neq f_{n-1}$, $\mathrm{n}\in\mathbb{N}$, there exists a continued fraction $b_0+K(a_m/b_m)$ with $\mathrm{n}^{\mathrm{th}}$ approximant $f_n$ for all $n$. This continued fraction is given by

$$c_0+\frac{c_1}{1}\,\frac{-c_2/c_1}{+1+c_2/c_1}+\cdots+\frac{-c_m/c_{m-1}}{1+c_m/c_{m-1}}+\cdots. \tag{2.9}$$

## 3. Main results

In this section, we provide a main method for construction of continued fraction based on a given power series. Then we establish a new continued fraction approximation for the Lugo and Euler–Mascheroni constants.

**Theorem 3.1.** Let $\{c_k\}$ be a sequence in $\mathbb{R}\setminus\{0\}$. Then for every $x\neq 0$,

$$\sum_{i=1}^{n}\frac{c_i}{x^i}=\frac{1}{x}\mathop{K}_{i=1}^{n}\frac{p_i x}{x+q_i}=\frac{1}{x}\frac{p_1 x}{x+\mathop{K}\limits_{i=2}^{n}\dfrac{p_i x}{x-p_i}}, \qquad \mathrm{n}\in\mathbb{N}, \tag{3.1}$$

where

$$p_1=c_1,\ q_1=0,$$
$$p_i=-\frac{c_i}{c_{i-1}},\ q_i=-p_i,\ i=2,3,\cdots,n.$$

**Proof.** Assume that

$$g_0(x)\neq\infty, \quad g_n(x)=\sum_{i=1}^{n}\frac{c_i}{x^i}, \quad \mathrm{n}\in\mathbb{N}, \quad x\neq 0, \tag{3.2}$$

The left-side of (3.1) is equal to $g_n(x)$.

Since

$$g_0(x)\neq\infty, \quad g_n(x)\neq g_{n-1}(x),\ \mathrm{n}\in\mathbb{N},$$

using Lemma 2.3,

$$g_n(x)=\sum_{i=1}^{n}\frac{c_i}{x^i}=\frac{\frac{c_1}{x}}{1}+\frac{-\frac{c_2}{c_1x}}{1+\frac{c_2}{c_1x}}+\frac{-\frac{c_3}{c_2x}}{1+\frac{c_3}{c_2x}}+\frac{-\frac{c_4}{c_3x}}{1+\frac{c_4}{c_3x}}+\cdots+\frac{-\frac{c_i}{c_{i-1}x}}{1+\frac{c_i}{c_{i-1}x}}+\cdots+\frac{-\frac{c_n}{c_{n-1}x}}{1+\frac{c_n}{c_{n-1}x}}$$

$$=\frac{1}{x}\frac{c_1}{1}+\frac{-\frac{c_2}{c_1x}}{1+\frac{c_2}{c_1x}}+\frac{-\frac{c_3}{c_2x}}{1+\frac{c_3}{c_2x}}+\frac{-\frac{c_4}{c_3x}}{1+\frac{c_4}{c_3x}}+\cdots+\frac{-\frac{c_i}{c_{i-1}x}}{1+\frac{c_i}{c_{i-1}x}}+\cdots+\frac{-\frac{c_n}{c_{n-1}x}}{1+\frac{c_n}{c_{n-1}x}}$$

$$=\frac{1}{x}\frac{c_1x}{x}+\frac{-\frac{c_2}{c_1}}{1+\frac{c_2}{c_1x}}+\frac{-\frac{c_3}{c_2x}}{1+\frac{c_3}{c_2x}}+\frac{-\frac{c_4}{c_3x}}{1+\frac{c_4}{c_3x}}+\cdots+\frac{-\frac{c_i}{c_{i-1}x}}{1+\frac{c_i}{c_{i-1}x}}+\cdots+\frac{-\frac{c_n}{c_{n-1}x}}{1+\frac{c_n}{c_{n-1}x}}$$

$$=\frac{1}{x}\frac{c_1x}{x}+\frac{-\frac{c_2}{c_1}x}{x+\frac{c_2}{c_1}}+\frac{-\frac{c_3}{c_2}}{1+\frac{c_3}{c_2x}}+\frac{-\frac{c_4}{c_3x}}{1+\frac{c_4}{c_3x}}+\cdots+\frac{-\frac{c_i}{c_{i-1}x}}{1+\frac{c_i}{c_{i-1}x}}+\cdots+\frac{-\frac{c_n}{c_{n-1}x}}{1+\frac{c_n}{c_{n-1}x}}$$

$$=\frac{1}{x}\frac{c_1x}{x}+\frac{-\frac{c_2}{c_1}x}{x+\frac{c_2}{c_1}}+\frac{-\frac{c_3}{c_2}x}{x+\frac{c_3}{c_2}}+\frac{-\frac{c_4}{c_3}}{1+\frac{c_4}{c_3x}}+\cdots+\frac{-\frac{c_i}{c_{i-1}x}}{1+\frac{c_i}{c_{i-1}x}}+\cdots+\frac{-\frac{c_n}{c_{n-1}x}}{1+\frac{c_n}{c_{n-1}x}}$$

$$=\cdots\ \cdots\ \cdots$$

$$=\frac{1}{x}\frac{c_1x}{x}+\frac{-\frac{c_2}{c_1}x}{x+\frac{c_2}{c_1}}+\frac{-\frac{c_3}{c_2}x}{x+\frac{c_3}{c_2}}+\frac{-\frac{c_4}{c_3}x}{x+\frac{c_4}{c_3}}+\cdots+\frac{-\frac{c_i}{c_{i-1}}x}{x+\frac{c_i}{c_{i-1}}}+\cdots+\frac{-\frac{c_n}{c_{n-1}}x}{x+\frac{c_n}{c_{n-1}}} \tag{3.3}$$

$$=\frac{1}{x}\frac{c_1x}{x+\underset{i=2}{\overset{n}{K}}\frac{-\frac{c_i}{c_{i-1}}x}{x+\frac{c_i}{c_{i-1}}}}=\frac{1}{x}\frac{c_1x}{x+0+\underset{i=2}{\overset{n}{K}}\frac{-\frac{c_i}{c_{i-1}}x}{x+\frac{c_i}{c_{i-1}}}}\ .$$

The middle expression of (3.1) is equal to

$$\frac{1}{x}\underset{i=1}{\overset{n}{K}}\frac{p_ix}{x+q_i}=\frac{1}{x}\frac{p_1x}{x+q_1+\underset{i=2}{\overset{n}{K}}\frac{p_ix}{x+q_i}},\qquad x\neq 0\,. \tag{3.4}$$

Thus,

$$p_1=c_1,\ q_1=0,$$

$$p_i=-\frac{c_i}{c_{i-1}},\ q_i=\frac{c_i}{c_{i-1}}=-p_i,\quad i=2,3,\cdots,n.$$

Then, it is obviously true that

$$\frac{1}{x}\mathop{K}_{i=1}^{n}\frac{p_i x}{x+q_i}=\frac{1}{x}\cfrac{p_1 x}{x+\mathop{K}\limits_{i=2}^{n}\cfrac{p_i x}{x-p_i}}, \qquad x\neq 0 . \tag{3.5}$$

The proof of Theorem 3.1 is complete.

**Theorem 3.2.** We have a new continued fraction approximation for the Lugo and Euler–Mascheroni constants:

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\mathop{K}_{i=1}^{\infty}\frac{p_i n}{n+q_i}=-\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\cfrac{p_1 n}{n+q_1+\cfrac{p_2 n}{n+q_2+\cfrac{p_3 n}{n+q_3+\ddots}}}\,(n\to\infty), \tag{3.6}$$

where

$$\begin{aligned}
p_1&=-\frac{5}{8},\ q_1=0,\\
p_2&=\frac{7}{30},\ q_2=-\frac{7}{30},\\
p_{2i-1}&=-\frac{(i-1)(2-2^{1-2i})B_{2i}}{i(2-2^{2-2i})B_{2(i-1)}}, q_{2i-1}=-p_{2i-1},\\
p_{2i}&=-\frac{2-2^{-2i}}{2-2^{1-2i}}, q_{2i}=-p_{2i}, \quad i=2,3,\cdots.
\end{aligned}$$

**Proof.** Let

$$\begin{aligned}
c_1&=-\frac{5}{8},\\
c_{2i}&=\frac{(2-2^{-2i})B_{2i}}{2i}, c_{2i+1}=\frac{(2-2^{-1-2i})B_{2(i+1)}}{2(i+1)},\ \ i=1,2,\cdots.
\end{aligned} \tag{3.7}$$

From (3.7) and Theorem 3.1,

$$\sum_{i=1}^{\infty}\frac{c_i}{n^i}=\frac{1}{n}\mathop{K}_{i=1}^{\infty}\frac{p_i n}{n+q_i}, \tag{3.8}$$

where

$$\begin{aligned}
p_1&=c_1=-\frac{5}{8},\ q_1=0,\\
p_2&=-\frac{c_2}{c_1}=\frac{7}{30},\ q_2=-p_2=-\frac{7}{30},\\
p_{2i-1}&=-\frac{c_{2i-1}}{c_{2i-2}}=-\frac{(i-1)(2-2^{1-2i})B_{2i}}{i(2-2^{2-2i})B_{2(i-1)}}, q_{2i-1}=-p_{2i-1},\\
p_{2i}&=-\frac{c_{2i}}{c_{2i-1}}=-\frac{2-2^{-2i}}{2-2^{1-2i}}, q_{2i}=-p_{2i}, \quad i=2,3,\cdots.
\end{aligned}$$

According to (2.4) and (3.8),

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\mathop{K}_{i=1}^{\infty}\frac{p_i n}{n+q_i}\,(n\to\infty). \tag{3.9}$$

Thus, our new continued fraction approximation can be obtained.

**Remark 3.1.** As you can see, our new continued fraction approximation for the Lugo and

Euler–Mascheroni constants is equal to Lugo's asymptotic formula but the expression is totally different.

From (3.1), we have another expression of (3.9) as follows:

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\frac{p_1 n}{n+\mathop{K}\limits_{i=2}^{\infty}\frac{p_i n}{n-p_i}}=-\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\cfrac{p_1 n}{n+\cfrac{p_2 n}{n-p_2+\cfrac{p_3 n}{n-p_3+\ddots}}}\ (n\to\infty), \quad (3.10)$$

where

$$p_1=-\frac{5}{8}, p_2=\frac{7}{30}, p_3=\frac{3}{28}, p_4=-\frac{31}{30}, p_5=\frac{15}{31}, \ \cdots.$$

For the convenience of readers, we rewrite.

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2+\frac{1}{n}\cfrac{-\frac{5}{8}n}{n+\cfrac{\frac{7}{30}n}{n-\frac{7}{30}+\cfrac{\frac{3}{28}n}{n-\frac{3}{28}+\ddots}}}\ (n\to\infty) \quad (3.11)$$

## 4. Conclusion

As mentioned above, in our investigation, we provide a generally applicable and useful method to construct continued fraction based on a given power series. Then we establish a new continued fraction approximation for the Lugo and Euler–Mascheroni constants. Especially, we analytically determine the coefficients of the Lugo's asymptotic formula and all parameters of the continued fraction by Bernoulli numbers.

## References


[1] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, Applied Mathematics Series, vol. 55, Nation Bureau of Standards, Dover, New York (1972).

[2] Chen , C. P.: Inequalities for the Lugo and Euler-Mascheroni constants, Appl. Math. Lett. 25, 787-792 (2012)

[3] Chen, C. P.: On the asymptotic expansions of the gamma function related to the Nemes, Gosper and Burnside formulas. Applied Mathematics and Computation. 276, 417-431 (2016)

[4] Chen, C.-P., Srivastava, H.M.: New representations for the Lugo and Euler–Mascheroni constants, Appl. Math. Lett. 24 (7), 1239–1244 (2011)

[5] Cuyt, A., Brevik Petersen, V., Verdonk, B., Waadeland, H., Jones, W.B.: Handbook of Continued Fractions for Special Functions, Springer, (2008)

[6] Lu, D.: A new quicker sequence convergent to Euler's constant. J. Number Theory 136, 320–329 (2014)

[7] Lu, D.: Some new improved classes of convergence towards Euler's Constant. Applied Mathematics and Computation 243, 24–32 (2014)

[8] Lu, D., Song, L.X., Yu, Y.: Some new continued fraction approximation of Euler's Constant. J. Number Theory 147, 69–80 (2015)
[9] Lugo, I.: An Euler-Maclaurin summation. Available at: http://www.math.upenn.edu/~mlugo /114-07C/euler-maclaurin.pdf.
[10] Mortici, C.: A new Stirling series as continued fraction, Numer. Algorithms 56(1), 17-26( 2011)
[11] Mortici, C.: A continued fraction approximation of the gamma function, J. Math. Anal. Appl. 402, 405-410 (2013)
[12] Srivastava, H.M., Choi, J.: Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001
[13] Wang, H. Z., Zhang, Q. L., Lu, D.: A quicker approximation of the gamma function towards the Windschitl's formula by continued fraction, Ramanujan J. (2018). https://doi.org/10.1007/s11139-017-9974-6.
[14] You, Xu.: Continued Fraction Approximation and Inequality of the Gamma Function, Results Math. 73, 37 (2018)
[15] You, Xu., Chen, D.R.: Some sharp continued fraction inequalities for the EulerMascheroni constant, J. Inequal. Appl. 2015, 308 (2015)