About an equation with radicals

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ABSTRACT

In this note we solve an equation with radicals and give two series for Pi

I. Introduction

Problem. solve the equation

$$f(x) = \frac{\sqrt{\sqrt{1+x^2}-1}}{\sqrt{2}\sqrt{1+x^2}+\sqrt{\sqrt{1+x^2}+1}} = 2\sqrt{2+\sqrt{3}}-\sqrt{3}-2 \tag{1}$$

Solution

$$z = \sqrt{\sqrt{1+x^2}-1} \ \wedge \ t = 2\sqrt{2+\sqrt{3}}-\sqrt{3}-2 \implies \frac{z}{\sqrt{2}\left(1+z^2\right)+\sqrt{2+z^2}} = t \tag{2}$$

$$(2) \implies 2t^2 z^3 - 2\sqrt{2}\, t z^2 + \left(1+3t^2\right) z - 2\sqrt{2}\, t = 0 \ \wedge \ x = \pm z\sqrt{2+z^2} \tag{3}$$

$$(3) \implies z = \begin{cases} z_1 = 0.4116\ldots \\ z_2 = 4.4010\ldots \\ z_3 = 5.9292\ldots \end{cases} \tag{4}$$

$$(3) \wedge (4) \implies x_1 = z_1\sqrt{2+z_1^2} = 0.6063\ldots,\ x_2 = -x_1\,, \\ x_3 = z_2\sqrt{2+z_2^2} = 20.344\ldots,\ x_4 = -x_3\,,\ x_5 = z_3\sqrt{2+z_3^2} = 36.142\ldots,\ x_6 = -x_5 \tag{5}$$

Verification of solutions

$$f(x_1) = f(x_2) = 2\sqrt{2+\sqrt{3}}-\sqrt{3}-2 \tag{6}$$

$$f(x_3) = f(x_4) = 2\sqrt{2+\sqrt{3}}-\sqrt{3}-2 \tag{7}$$

$$f(x_5) = f(x_6) \neq 2\sqrt{2+\sqrt{3}}-\sqrt{3}-2 \tag{8}$$

The set of solutions is:

$$S = \{x_1, x_2, x_3, x_4\} \tag{9}$$

II. Pi formula

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \tag{10}$$

Entry 1.

$$p = \frac{3 + 5\sqrt{2} + 4\sqrt{3} + 3\sqrt{6}}{3} \tag{11}$$

$$q = \frac{50 + 28\sqrt{2} + 18\sqrt{3} + 18\sqrt{6}}{27} \tag{12}$$

$$w = \frac{2 + 2\sqrt{2} + 2\sqrt{3} + \sqrt{6}}{3} - \sqrt[3]{q + p\sqrt[3]{q + p\sqrt[3]{q + \ldots}}} \tag{13}$$

$$s = w\sqrt{2 + w^2} = 0.606315861219\ldots,\ (s = x_1) \tag{14}$$

$$\pi = 6\sqrt{2}\sqrt{\sqrt{1 + s^2} + 1}\sum_{n=1}^{\infty}(-1)^{n-1}\binom{4n-2}{2n-1}H_{2n-1}2^{-4n+2}s^{2n-1} + 6\sqrt{2}\sqrt{\sqrt{1 + s^2} - 1}\sum_{n=1}^{\infty}(-1)^{n-1}\binom{4n}{2n}H_{2n}2^{-4n}s^{2n} \tag{15}$$

where $\mathrm{H_n} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}}$ is the harmonic number.

III. Endnote

Entry 2.

$$\pi = 12\sqrt{2}\sum_{n=1}^{\infty}\binom{2n}{n}2^{-2n}H_n z^n w(n) \tag{16}$$

where

$$z = \alpha - \left(q + p\left(q + p\left(q + p(q + \ldots)^3\right)^3\right)^3\right) = 0.2031266828275773\ldots \tag{17}$$

$$w(n) = \mathrm{Im}\left((1 + i)^n\left(\sqrt{1 - z + \sqrt{(1 - z)^2 + z^2}} - i\sqrt{-1 + z + \sqrt{(1 - z)^2 + z^2}}\right)\right),\ i = \sqrt{-1} \tag{18}$$

$$\alpha = \frac{\mathrm{A1}}{\mathrm{A2}} = 34.21058714031\ldots,\quad p = \frac{\mathrm{B1}}{\mathrm{B2}} = 0.00013171097,\quad q = \frac{\mathrm{C1}}{\mathrm{C2}} = 28.82728379282\ldots \tag{19}$$

$$\begin{gathered}\mathrm{A1} = -8 + \sqrt{12 - 6\sqrt{2}} + \sqrt{6 - 3\sqrt{2}} + \sqrt{4 - 2\sqrt{2}} + \\ \sqrt{2 - \sqrt{2}} - \sqrt{2 + \sqrt{2}} + \sqrt{4 + 2\sqrt{2}} + \sqrt{6 + 3\sqrt{2}} - \sqrt{12 + 6\sqrt{2}}\end{gathered} \tag{20}$$

$$\mathrm{A2} = 3\left(-4 + \sqrt{12 - 6\sqrt{2}} + \sqrt{6 - 3\sqrt{2}} - \sqrt{2 + \sqrt{2}} + \sqrt{4 + 2\sqrt{2}}\right) \tag{21}$$

$$B1 = 3\left(\sqrt{4-2\sqrt{2+\sqrt{2}}} + \sqrt{2-\sqrt{2+\sqrt{2}}} + \sqrt{\left(2+\sqrt{2}\right)\left(2-\sqrt{2+\sqrt{2}}\right)} - \sqrt{3\left(2+\sqrt{2+\sqrt{2}}\right)} - \sqrt{6\left(2+\sqrt{2+\sqrt{2}}\right)} + \sqrt{3\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)}\right)^4$$

$$B2 = 4\left(-4-\sqrt{2}+\sqrt{6}+8\sqrt{12-6\sqrt{2}}+8\sqrt{6-3\sqrt{2}}-4\sqrt{4-2\sqrt{2}} - 4\sqrt{2-\sqrt{2}}-8\sqrt{2+\sqrt{2}}+8\sqrt{2\left(2+\sqrt{2}\right)}-4\sqrt{3\left(2+\sqrt{2}\right)}+4\sqrt{6\left(2+\sqrt{2}\right)}\right) \tag{23}$$

$$C1 = -4\left(-104-12\sqrt{2}-24\sqrt{6}+9\sqrt{12-6\sqrt{2}}+9\sqrt{6-3\sqrt{2}}+45\sqrt{4-2\sqrt{2}} + 43\sqrt{2-\sqrt{2}}-7\sqrt{2+\sqrt{2}}+9\sqrt{2\left(2+\sqrt{2}\right)}+45\sqrt{3\left(2+\sqrt{2}\right)}-45\sqrt{6\left(2+\sqrt{2}\right)}\right) \tag{24}$$

$$C2 = 9\left(-4-\sqrt{2}+\sqrt{6}+8\sqrt{12-6\sqrt{2}}+8\sqrt{6-3\sqrt{2}}-4\sqrt{4-2\sqrt{2}} - 4\sqrt{2-\sqrt{2}}-8\sqrt{2+\sqrt{2}}+8\sqrt{2\left(2+\sqrt{2}\right)}-4\sqrt{3\left(2+\sqrt{2}\right)}+4\sqrt{6\left(2+\sqrt{2}\right)}\right) \left(\sqrt{4-2\sqrt{2+\sqrt{2}}} + \sqrt{2-\sqrt{2+\sqrt{2}}} + \sqrt{\left(2+\sqrt{2}\right)\left(2-\sqrt{2+\sqrt{2}}\right)} - \sqrt{3\left(2+\sqrt{2+\sqrt{2}}\right)} - \sqrt{6\left(2+\sqrt{2+\sqrt{2}}\right)} + \sqrt{3\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)}\right)^2 \tag{25}$$

Remark: $i = \sqrt{-1}$, $a, b \in \mathbb{R}$, $\mathrm{Im}\,(a + b\,i) = b$

IV. References

[1] K.R. Manning, A history of extraneous solutions, The Mathematics Teacher 63,1970.

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[4] H. Chen, Interesting Series Associated with Central Binomial Coefficients, Catalan Numbers and Harmonic Numbers. Journal of Integer Sequences, 2016, Article 16.1.5.