How to Solve Diophantine Equations

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Abstract: We present in this article a general approach (in the form of recommendations and guidelines) for tackling Diophantine equation problems (whether single equations or systems of simultaneous equations). The article should be useful in particular to young "mathematicians" dealing mostly with Diophantine equations at elementary level of number theory (noting that familiarity with elementary number theory is generally required).

Keywords: Diophantine equation, system of Diophantine equations, number theory.[1]

[1] All symbols and abbreviations in this paper are defined in § Nomenclature.

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1 Introduction

The subject of Diophantine equations is one of the oldest and richest mathematical branches in number theory (which is one of the oldest and richest branches, if not the oldest and richest, of all mathematics). The roots of this subject can be traced back to the ancient Babylonians who dealt with Pythagorean triple problems (which are essentially Diophantine equations).[2]

The label of these equations as "Diophantine equations" is another indication to their ancient roots since it originates from the name of the ancient Greek-Egyptian mathematician (i.e. Diophantus of Alexandria) who lived in the third century and who documented in his books problems which are modeled by this type of equations. In fact, being a Greek-Egyptian mathematician should also suggest an oriental origin of these types of equations and the mathematical branch that is based on them. Apparently, ancient Hindu mathematicians have also dealt with this type of equations and the methods of their solution as early as the fifth or sixth century BC.

Despite their apparent simplicity, Diophantine equations are usually difficult (if not impossible) to solve. In fact, Diophantine problems (represented mostly in solving Diophantine equations and systems of such equations) are generally more difficult to tackle and solve than their corresponding ordinary versions. This is because the demand for the solutions (and answers in general) to be integers imposes extra requirements and conditions and hence it usually complicates the process and methods of solution. In this context, it is worth noting that one of the most famous open problems in mathematics (which waited several centuries of mathematical development and required huge efforts from many prominent mathematicians before it was finally solved in the mid 1990's) is Fermat's last theorem which is a Diophantine equation problem. Also, there are still many unsolved open problems about the subject of Diophantine equations or related to it.

[2] In fact, the credit for the "Pythagorean triples" should be attributed in the first place to the Babylonians (and possibly other civilizations of ancient Mesopotamia) who recorded and used these triples in their calculations (mainly for the purpose of surveying farm lands) more than one millennium before Pythagoras and the Greek Pythagorean school. There is also evidence (or indication) that the ancient Egyptians used these triples in their calculations for similar reasons and purposes as the Babylonians (and possibly even in the engineering of their marvelous constructions like the great Pyramids). So, it should be more fair to call these triples something like "Babylonian triples" or "Babylonian-Pythagorean triples". For more details about this historical issue, the reader should refer to the literature of the history of mathematics with special attention to the literature about the Babylonian clay tablets related to the mathematics of ancient Mesopotamia (e.g. Plimpton 322 and Si.427 tablets) which are investigated by a number of archaeologists, mathematicians and historians of science (such as Dr. Daniel Mansfield of the University of New South Wales; see for instance [1]).

In this article we propose a general strategy for tackling and solving Diophantine equations. This strategy is presented in the form of a list of recommendations and guidelines that can (and should) be used as a reference in tackling Diophantine problems. In fact, most of these recommendations and guidelines are based on (and extracted from) our personal experience in solving (as well as in reading) Diophantine equation problems and the methods (or techniques or tricks or ...) which are commonly used in their solution (see for example [2, 3]). The structure of this paper is simply based on the aforementioned list where each one of the following sections is essentially based on one of the recommendations and guidelines (with consideration of their order in part, i.e. when order is relevant or required).[3]

We should finally note (before we go through this investigation) that a Diophantine equation (as well as a system of equations) may have no solution, or a single solution, or multiple solutions (whether finitely many or infinitely many). So, "solving" (or "finding the solution") of a Diophantine equation (or system) should mean *proving* (by an irrefutable logical/mathematical argument) that there is no solution (i.e. when there is no solution) or *finding* all the solutions (either explicitly or through a sort of closed form formula or formulae) with an incontestable argument that there are no other solutions. So, a Diophantine problem (whether equation or system) is not solved, for instance, by finding a number of solutions (e.g. through inspection or through computational search) even if we know for sure that there are no other solutions. We should also note that a solution of a system of Diophantine equations is a solution that satisfies all the equations in the system simultaneously (i.e. the set of solutions of a system of Diophantine equations is the intersection of the sets of solutions of its individual equations). So, these criteria and considerations about "solving" Diophantine equations and systems should apply throughout this paper.

We should also note that there are two main methods for solving systems of Diophantine equations in number theory. The first is based on using the traditional methods of solving systems of multivariate equations (as investigated in algebra and linear algebra for instance) such as by substitution or comparison or use of the techniques of matrices, and the second is by solving the individual equations separately (either by the general methods of algebra or by the special methods and techniques of number theory) and selecting the

[3] However, we should note that the order of the following sections (and hence the recommendations they present) does not necessarily reflect the required order of these recommendations in practical situations noting that the order generally depends on the type of the Diophantine problem and its characteristic features. Anyway, commonsense and wise judgment should always be the resort that determines the order (as well as almost everything else).

solutions that satisfy the system as a whole (i.e. by accepting only the solutions which are common to all the equations).[4] In this regard we remind the reader of what we already said, that is: the set of solutions of a system of equations is the intersection of the sets of solutions of its individual equations. As a result, a system of equations is solvable only if its individual equations are solvable, although the converse is not true in general. Accordingly, a system of equations has no solution if some of its equations have no solution, but a system may not have a solution even though all its individual equations have solutions (i.e. when the intersection of these solutions is the empty set).

[4] In this context, we should pay special attention to the method of having (or obtaining) the solution of one equation and test it on the other equations in the system. In brief, if the system of Diophantine equations contains an equation whose solution is known (or can be obtained easily or more easily) then the best approach for solving the system is to test the solutions of that equation on the other equations where only the common solutions (if any) to all equations are accepted. This is particularly true when some of the equations in the (non-linear) system are linear. This approach usually saves considerable amounts of time and effort in trying to solve the system by other methods. In fact, in some cases this can be the only viable method for solving the system.

2 Initial Sensibility Checks

The first recommendation is to conduct initial (and basic) sensibility checks to assess the sensibility of the equation quickly (by inspecting its general characteristics) to see if it is possible to have a solution or not. These initial checks may also reveal the obvious solutions of the equation easily without effort or use of any complicated treatment. In the following subsections we outline some of the most common initial sensibility checks (where we dedicate the last subsection to our "final thought" about this issue).

2.1 Parity Checks

Parity checks should be regarded as the first item in the list of sensibility checks. This is due to its simplicity and intuitivity. For example, if we are asked to find the general solution of the Diophantine equation:

$$x^4 + 4y^3 - 7x^2 - 12y + 7 = 0 \qquad (x, y \in \mathbb{Z})$$

then before we try to solve this equation by using the familiar rules and traditional methods of solving polynomial Diophantine equations (in two variables) we should simply check the parity of this polynomial, and hence we can easily conclude (by checking the parity) that this equation has no solution because the polynomial is always odd and hence it cannot be equal to 0 which is even.

Similarly, the Diophantine equation:

$$x^9 - x = y^5 - 13y^2 + 1 \qquad (x, y \in \mathbb{Z})$$

can be "solved" with no effort by noting that the left hand side is always even while the right hand side is always odd and hence this equation has no solution in integers (as it is supposed to be a Diophantine equation).

Another example is the exponential equation:

$$17^x - 13^y = 19^z \qquad (x, y, z \in \mathbb{N}^0)$$

which obviously has no solution because the LHS is even while the RHS is odd.

Parity checks can also reduce the possibilities that to be considered (or the domain of the problem). For example, the equation:

$$18^x + 16^y = 19^z \qquad (x, y, z \in \mathbb{N}^0)$$

can have a solution (in principle) but because of parity considerations any potential solution must have either $x = 0$ (and $y \neq 0$) or $y = 0$ (and $x \neq 0$). So, we have only these possibilities to consider which by simple inspection should lead to the only solution, i.e. $(x, y, z) = (1, 0, 1)$.

2.2 Primality and Composity Checks

For example, if we conduct an initial primality check on the following Diophantine equation:

$$5x^2 + 125y^3 = 4973 \qquad (x, y \in \mathbb{Z})$$

then it should be fairly obvious that this equation has no solution because 4973 is prime while $5x^2 + 125y^3 = 5(x^2 + 25y^3)$ and hence they cannot be equal considering their prime factorization.[5]

Similarly, the equation:

$$6x^3 - 19x^2y + 19xy^2 - 6y^3 = p \qquad (x, y \in \mathbb{Z} \text{ and } p \in \mathbb{P})$$

has no solution because the left hand side cannot be prime (considering its factorization) and hence this Diophantine equation has no solution.[6]

2.3 Sign and Magnitude Checks

For example, the Diophantine equation:

$$x^2 + y^2 + 1 = 0 \qquad (x, y \in \mathbb{Z})$$

has obviously no solution because $x^2 + y^2$ cannot be negative and hence when it is added to 1 the result cannot be zero.

Similarly, the equation:

$$x^4 + y^2 + z^6 = 0 \qquad (x, y, z \in \mathbb{Z})$$

has the obvious (and only) trivial solution (i.e. $x = y = z = 0$) because any sum of even natural powers of integers must be a positive natural number unless all the integers are 0.

[5] Alternatively, we may say: because 4973 is prime while $5x^2 + 125y^3 = 5(x^2 + 25y^3)$ which is either composite or equal to 5 and hence they cannot be equal.

[6] The LHS can be factorized as $(2x - 3y)(3x - 2y)(x - y)$. Now, a prime number p can be factorized as a product of three integer factors only in 3 different ways, i.e.

$$p = (1)(-1)(-p) = (1)(1)(p) = (-1)(-1)(p)$$

It can be shown that none of these ways are applicable in this case and hence the LHS cannot represent a prime number.

Also, it is fairly obvious that the equation:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=5 \qquad\qquad (x,y,z\in\mathbb{Z} \text{ and } xyz\neq 0)$$

has no solution in integers because of a magnitude issue (i.e. the left hand side cannot be greater than 3).

This also applies to the equation:

$$12^x+23^y=199 \qquad\qquad (x,y\in\mathbb{N}^0)$$

because the left hand side is either less than 199 or greater than 199 (i.e. there is no combination of x,y that produces 199 where we can reach this conclusion easily by just checking the low-values of x,y).

2.4 Simple Divisibility Checks

For example, it should be fairly obvious that the equation:

$$x^4-x^2+y^4-y^2=18 \qquad\qquad (x,y\in\mathbb{Z})$$

has no solution because the left hand side is divisible by 4 $\left[\text{noting that } x^4-x^2=(x^2-x)(x^2+x) \text{ where both factors are even and this similarly applies to } y^4-y^2\right]$ while the right hand side is not divisible by 4.

2.5 Simple Modular Arithmetic Checks

For example, it should be fairly obvious (to someone with modest experience in solving Diophantine equations) that the Diophantine equation:

$$15x^2+6y^2=12 \qquad\qquad (x,y\in\mathbb{Z})$$

has no solution because by a simple modularity inspection (i.e. via reducing the equation in modulo 5) we find that this equation implies $y^2\overset{5}{=}2$ which obviously has no solution (because 2 is a quadratic non-residue of 5) and hence the original Diophantine equation is not solvable. Also see § 6.2.4 and § 7.

It is worth noting that we can consider parity checks (which we investigated earlier in § 2.1) as an instance of simple modularity (or modular arithmetic) checks. In fact, we can consider parity checks as the simplest modularity checks (since parity checks are based on the modular arithmetic of 2 which is the least modulo in modular arithmetic). However, parity checks (unlike common modularity checks) are not limited to explicit modularity inspection and checks, and hence from this perspective we may consider parity checks as more general than other modularity checks.

2.6 Final Thought about Initial Sensibility Checks

It is important to note that many Diophantine equation problems do not need in their solution more than an informed inspection based on these initial and simple checks and hence it is worthwhile to spend a few minutes on doing this sort of initial inspection and tests which can save a considerable amount of time and effort in trying to solve the given problem by the use of sophisticated approaches and techniques (which may or may not lead to the required result). So in brief, an initial and systematic inspection using general rules (such as the rules of parity, primality, sign, divisibility, etc.) can save a lot of time trying to solve an equation that has no solution or has an obvious solution and hence it does not require any effort to solve.

It should be obvious that conducting initial sensibility checks as a first step in dealing with Diophantine problems applies not only to single Diophantine equations but also to systems of Diophantine equations. So, if a system contains a non-solvable equation then the system is not solvable. Therefore, it is worthwhile to inspect the individual equations of the system (to check if they are solvable or not) before trying to solve the system. For example, if we are given the following system:

$$x^2 + y^3 - z^4 = 3 \qquad \text{and} \qquad 2x + y^2 - 3y - 6z^2 = 75 \qquad (x, y, z \in \mathbb{Z})$$

then a quick initial inspection should reveal that the second equation has no solution (due to parity violation) and hence the system has no solution (with no need for inspecting the first equation or the system as a whole).

It is also worthwhile to inspect the characteristics of the system as a whole (i.e. not only its individual equations separately) to see if it is sensible for the system to have a solution or not. For example, if we are given the following system of Diophantine equations:

$$x^2 + y^8 + z^6 = 0 \qquad \text{and} \qquad x^4 + y^2 - 17 = 0 \qquad (x, y, z \in \mathbb{Z})$$

then a quick initial inspection should reveal that the first equation has only the trivial solution (i.e. $x = y = z = 0$), while the second equation can have only non-trivial solutions (i.e. it cannot accept the trivial solution). This means that the two equations cannot have a common solution and hence the system has no solution.

Initial inspection to the system as a whole should also reveal that the following system (where $x, y, z \in \mathbb{Z}$):

$$x^6 + y^2 - z^3 = 0 \qquad \text{and} \qquad x^2 + y^4 + z + 1 = 0 \qquad \text{and} \qquad x + y + z + 1 = 0$$

has no solution because if the first equation has a solution then z must be non-negative, while if the second equation has a solution then z must be negative. So, the two equations cannot have a common solution and hence this system cannot have a solution.

3 Graphic Inspection

Many systems of Diophantine equations can be solved graphically (or at least graphic investigation of these systems can help in finding their solutions). This is particularly true when we deal with 2-variable systems of Diophantine equations of various types (such as polynomials and exponentials). So, it is recommended to use graphic tools (when possible and applicable) for initial inspection of systems of Diophantine equations by plotting the functions representing these equations on the same graph to see if and where they have points of intersection. As indicated, the use of these tools can lead to the final solution of the problem without further action.

In simple cases, "graphic inspection" may not require actual plotting of graphs, i.e. we just "plot" the graphs mentally or use graphic arguments and considerations to reach the final solution. For example, the following system of Diophantine equations:

$$x^2 - 2x + 4 + y = 0 \qquad \text{and} \qquad x^2 + y^2 + 6x - 10y + 30 = 0$$

can be solved "graphically" by noting that the first equation represents a parabola while the second equation represents a circle, and hence the problem can be solved "graphically" with no need to make any plot. So, if we put these equations in their standard forms (so that we can easily identify the shape and position of the graphs they represent) then we have:

$$y + 3 = -(x - 1)^2 \qquad \text{and} \qquad (x + 3)^2 + (y - 5)^2 = 4$$

Now it is obvious that the first equation represents a parabola which concaves down with vertex at $(x, y) = (1, -3)$ while the second equation represents a circle with radius 2 and center at $(x, y) = (-3, 5)$ and hence they cannot have a common point. Therefore, this system obviously has no solution.

Graphic inspection can also be useful in solving individual Diophantine equations (i.e. not only systems of such equations) since graphic inspection can provide an insight about the nature of the equation and if it can/cannot have a solution (or at least if it has an obvious solution, e.g. when the graphic plot can show such a solution).

4 Manipulations and Transformations

We should always consider manipulating the given Diophantine equation (algebraically and non-algebraically) or/and transforming its variables to put the equation in a "more friendly" (or more recognizable or more familiar) form, e.g. by being of a familiar standard form whose solution can be obtained more easily or by simplifying its form and hence making its analysis and investigation more easy. Simple (algebraic and non-algebraic) manipulations include for instance:

• Changing the symbols to improve the look of the equation and remove potential sources of confusion and ambiguity.

• Moving terms from one side to the other (e.g. to separate the variables which could make the equation easier to analyze and tackle).

• Multiplying/dividing the two sides of the equation by a constant (e.g. by -1 to change signs or by a constant in a denominator to remove a fraction).

• Multiplying/dividing the two sides of the equation by a variable or variables (but we should remember that such manipulations could introduce or eliminate some solutions and hence all the obtained solutions after this type of manipulation should be tested on the original equation as well as considering other potential solutions).

• Raising to integer powers (e.g. by squaring and cubing) to remove roots. However, this sort of manipulation can affect the solutions (e.g. by introducing foreign solutions), and hence we recommend again testing the obtained solutions after this type of manipulation on the original equation.

• Raising to fractional powers which is equivalent to taking roots (whether integer or fractional roots). Again, this sort of manipulation can affect the solutions and hence the obtained solutions after this type of manipulation should be tested on the original equation.

• Grouping terms of certain types within brackets and parentheses. For example, by grouping the terms according to their variables (e.g. those involving x versus those involving y) the equation can become easier to recognize, categorize and analyze. Similarly, grouping the terms according to their degree (e.g. linear, squared, cubic, etc.) can have a similar beneficial effect.

• Completing squares and hence reducing the size of the equation and possibly putting it in a certain standard form whose solution can be obtained more easily. For example, the Diophantine equation:

$$x^2 + y^2 - 6x + 12y - 36 = 0$$

can be put in the following quadratic form:

$$(x-3)^2 + (y+6)^2 = 9^2$$

by completing the squares. This new form is much simpler to analyze than the original equation form and hence it can be solved more easily. For instance, the new form will naturally indicate the restrictions $0 \le (x-3)^2 \le 9^2$ and $0 \le (y+6)^2 \le 9^2$ and hence to find the solutions we need no more than testing the few possibilities of the values of x and y obtained by considering these restrictions.

We should also consider transforming the variables of the equation (with or without manipulations which we dealt with already). Transformation of variables can greatly simplify the given equation and hence make it easier to analyze and solve. It can also put it in a more recognizable (and possibly standard) form (such as Pell's equation form; see § 6.2.3) whose solution can be obtained more easily.

For example, with some simple manipulations the equation:

$$4x^2 - 6y^2 + 12x + 108y - 478 = 0$$

can be transformed from its (rather messy) form to the tidy form:

$$X^2 - 6Y^2 = 1$$

where $X = 2x+3$ and $Y = y-9$ and hence it can be solved more easily as a Pell equation where the final solutions are obtained by the reverse transformations $\big[$i.e. $x = (X-3)/2$ and $y = (Y+9)\big]$ subject to certain conditions.

This similarly applies to the equation:

$$9x^2 - 325y^2 - 42x - 130y + 35 = 0$$

which can be transformed to the tidy form:

$$X^2 - 13Y^2 = 1$$

where $X = 3x - 7$ and $Y = 5y + 1$ (noting that the Pell solutions will not lead to solutions to the original equation because the reverse transformations do not produce integer solutions).

We may also employ a single-variable transformation. For example, the equation:

$$10x^2 + 2x - 8y^2 = 0$$

can be transformed to:

$$X^2 - 80y^2 = 1$$

where $X = 10x + 1$ and hence it can be solved more easily where the solutions of the original equation are obtained by the reverse transformation, i.e. $x = (X - 1)/10$ subject to certain conditions.

More simple transformations (such as by changing the sign of some or all variables) may also be used to improve the shape and form of the equation or to put it in a more solvable form, e.g. by being similar to an already-solved problem or by being in a certain standard form whose solution can be obtained readily (see for instance the examples in § 9).

So in brief, manipulations and transformations can generally improve the "look and feel" of the equation (among other beneficial effects) and hence they usually make it tidier, simpler and easier to analyze and solve.

5 Initial Computational Investigation

Initial investigation of Diophantine problems by the use of computational tools (such as computer codes or spreadsheets or software packages) should provide an initial impression and insight about the nature of the expected (and sought-after) solutions. In fact, initial computational investigation can be a great help in identifying and producing a theoretical and general argument (or proof or formulation or ...) that solves the problem completely and unequivocally.

For example, if we conduct an initial computational investigation on the Diophantine equation:

$$4x^2 + 4x - 15 - y^3 = 0$$

by a computer code (which loops, for instance, over all integers x, y between -10000 and $+10000$) then we should get no solution in this range and this should suggest that this equation has no solution. So, if we now inspect this equation further (in the light of this suggestion) then we may note that completing the square in x will lead to the equation $y^3 = (2x + 1)^2 - 16$ which can be factorized as $y^3 = (2x - 3)(2x + 5)$. Further reasoning (based on this type of factorization) should lead to the conclusion that this equation cannot have a solution.

A similar initial computational investigation on the Diophantine equation:

$$x^3 + y^3 + z^3 = 58$$

should also fail to find a solution in the given range and this should suggest that this equation has no solution. Further inspection and analysis (based, for instance, on modular arithmetic) should produce a conclusive argument that this equation has no solution (e.g. if we reduce this equation in modulo 9 then we get the congruence equation:

$$x^3 + y^3 + z^3 \overset{9}{=} 4$$

which has no solution because the sum of three cubes cannot be congruent to 4 modulo 9 and hence the original Diophantine equation cannot have a solution).

So, the insight obtained by these initial computational investigations provides a great help in solving these Diophantine equations (i.e. by producing a conclusive argument, based on the aforementioned factorization or modular arithmetic analysis, that these equations have no solution).

Similarly, if we conduct an initial computational investigation on the following Diophantine equation in a certain range (say as before):

$$xy + yz = xyz$$

then we will find out that all the solutions that we get in the given range have one of 3 forms: $y = 0$ and x and z are arbitrary, $x = z = 0$ and y is arbitrary, and $x = z = 2$ and y is arbitrary. Further inspection and analysis (based on the insight provided by the patterns of the obtained solutions) should lead to a conclusive argument that this equation actually has only 3 types of solution, i.e. $(x, y, z) = (m, 0, k)$, $(0, n, 0)$ and $(2, n, 2)$ where $m, n, k \in \mathbb{Z}$. So, thanks to the insight obtained by this initial and easy computational investigation we were able to produce a conclusive logical/mathematical argument that determines all the solutions of this equation unequivocally. In fact, without such computational investigations solving equations like this can be much more difficult.

There are many other examples that demonstrate the aid provided by simple initial computational investigation in tackling Diophantine equation problems and finding their solutions. Many of the examples that will be given in the future (for various purposes) are based on (and initiated by) such computational investigation effort.

As indicated earlier (see the paragraph before the last of § 1), computational investigation on its own cannot provide a solution to Diophantine problems even if this investigation leads to finding all the actual solutions. So, it should be obvious that "Initial Computational Investigation" (which is the title of this section) is no more than an initial step in the search for solution (i.e. proving that there is no solution or finding all the solutions with a conclusive logical/mathematical argument that there are no other solutions).

6 Classification and Recalling Standard Methods

The obvious next recommendation (assuming the problem has not been solved so far) is to classify the problem such as being linear or non-linear, exponential or polynomial (involving quadratic or/and cubic or $\ldots$), 2- or 3- $\cdots$ or n-variable problem, and so on. This should obviously be associated with (or followed by) recalling the standard methods of solution for the specific type (as identified according to its classification). In this regard, we recommend using previously-solved similar problems (related to the identified specific type) as prototypes and models to see if it is possible to apply their methods of solution to the problem at hand (also see § 9). In fact, in some cases solving the problem at hand may require no more than copying and pasting the solution of a previously-solved problem with some modifications and adaptations to reflect the specific characteristics of the problem at hand.

In the following subsections we investigate briefly some common types of Diophantine problems and some standard methods of their solution.

6.1 Linear Equations

Solving linear Diophantine equations should be straightforward in general because there are closed form formulae for their solutions (at least for the 2- and 3-variable equations). Yes, sometimes the statement of the problem requires an investigation (or an insight) about how to model it by a linear Diophantine equation or system of equations, but this is another story which is related to modeling rather than solving a modeled (i.e. already-formulated) problem which is what our paper is about.

In this regard, we should remember that a linear Diophantine equation has either no solution or infinitely many solutions. However, mathematical restrictions on the domain or range of solution (e.g. being in natural numbers, possibly within a given range, instead of being in integers) should generally reduce the solutions (although it may or may not affect their infinitude). Physical restrictions on the nature of specific problem may also reduce the solutions by making the number of solutions finite or even zero (i.e. when the physical restrictions cannot be satisfied by the mathematical formulation and hence the equation or system cannot have a solution within the given physical restrictions).[7]

It is also useful to remember that a linear Diophantine equation is always solvable (and hence it has infinitely many solutions from a mathematical viewpoint) when it is

[7] The common example of this is when the physical restrictions require positive values while the mathematical formulation leads to negative values.

homogeneous and when the greatest common divisor of its coefficients is 1.

Regarding systems of linear Diophantine equations, they generally can be solved by the well-known methods of linear algebra (with the restriction on the solutions to be integers) which the interested reader should refer to in the wide mathematical literature. However, these systems may also be solved by solving their individual equations (by the usual techniques and methods of number theory) with taking the intersection of their solutions (which could be the empty set).[8] For example, if we solve the following system:

$$15x + 10y + 30z = 41 \qquad \text{and} \qquad 22x - 21y + 8z = 5 \qquad \text{and} \qquad x + 19y - 39z = 73$$

by the familiar methods of linear algebra we get no integer solution and hence there is no solution to this system in $\mathbb{Z}$. This result can also be reached (more simply and directly) by noting that the left hand side of the first equation is 0 (modulo 5) while its right hand side is 1 (modulo 5) and hence this equation (as well as the system) has no solution.

6.2 Non-Linear Polynomial Equations

There is no standard method (whether single or multiple) for solving non-linear polynomial Diophantine equations as they come in many different shapes and forms (varying, for instance, in the number of variables, the highest degree of each variable, whether they contain mixed-variables terms or not, and so on). However, there are a number of recommendations and considerations (including methods and techniques) that should be remembered when dealing with non-linear polynomial Diophantine equations. Some of the most common of these recommendations and considerations are outlined in the following sub-subsections.

6.2.1 Fermat's Last Theorem

We should remember Fermat's last theorem when we deal with polynomial Diophantine equations in 2 or 3 variables with no mixed-variables terms. This theorem states: no natural numbers a, b, c satisfy the equation $a^n + b^n = c^n$ for any $n \in \mathbb{N}$ greater than 2. Hence, this theorem is about eliminating the possibility of solutions of certain type. In this regard it is useful (and possibly important) to keep the following points in mind when dealing with such equations:

1. Although this theorem is about "natural numbers" it also extends to negative integers because if n is even then the power does not distinguish between positive and nega-

[8] This issue has been outlined earlier in this paper (see the last paragraph of § 1).

tive, while if n is odd then the difference between the negatives and positives is just a multiplicative factor of -1 and hence it does not affect solvability.[9]

2. This theorem is about "natural numbers" (as well as their negatives) and hence the equation $a^n + b^n = c^n$ is generally solvable when we extend its domain to include 0. In other words, this equation generally accepts "trivial" solutions in the extended sense of trivial, i.e. when some (and not necessarily all) variables are 0. This is important to remember to avoid the mistake (which some people commit) that this equation has only the *trivial* solution (i.e. $a = b = c = 0$) which is not true. For example, $a^3 + b^3 = c^3$ has solutions like $(a, b, c) = (0, 1, 1)$ or $(2, 0, 2)$ which are not *trivial* although they are "trivial" in the aforementioned extended sense.
3. Fermat's last theorem should also be remembered when dealing with polynomial Diophantine equations in 2 variables of the above type, i.e. when dealing with equations of the form $a^n + b^n = m$ or $a^n + b^n + m = 0$ where m is a given integer (e.g. 27). In other words, we should inspect m to see if it is an n^{th} power of an integer and hence the equation can be subject to Fermat's last theorem.
4. The above standard form (i.e. $a^n + b^n = c^n$) of the equation that is subject to Fermat's last theorem can be disguised and hence we should always inspect the possibility of manipulating or transforming a "disguised Fermat's last theorem equation" to put it in its standard form (i.e. $a^n + b^n = c^n$). For example, the following equations are "disguised Fermat's last theorem equations":

$$x^3 + y^6 = z^9 \qquad x^3 - y^3 = z^3 \qquad x^5 - 7776 = z^5 \qquad x^6 - y^3 + 216z^3 = 0$$

because these equations can be put in the following standard forms:

$$x^3 + Y^3 = Z^3 \qquad y^3 + z^3 = x^3 \qquad z^5 + 6^5 = x^5 \qquad X^3 + \zeta^3 = y^3$$

where $Y = y^2$, $Z = z^3$, $X = x^2$ and $\zeta = 6z$. Other forms of manipulations and transformations (usually more complicated than the above) can put "disguised Fermat's last theorem equations" in their standard form. So, we should always be vigilant and resourceful (by using manipulations and transformations) so that we can make use (if possible) of Fermat's last theorem when we deal with equations of such types.

[9] Some cases of mixed positive and negative integers when n is odd can be dealt with by manipulations (as will be outlined in point 4).

6.2.2 Pythagorean Triple Rules

We should also remember the Pythagorean triple rules and theorems and the equations representing them when dealing with quadratic equations in 2 or 3 variables with no mixed-variables terms. In this regard, we should consider manipulating the equation (if necessary) to put it in a standard Pythagorean triple equation form (e.g. by completing the squares or/and by transforming the variables; see § 4).

For example, if we manipulate the Diophantine equation:

$$4x^2 + 16y^2 - z^2 + 4x - 24y + 10 = 0$$

then we get:

$$(2x + 1)^2 + (4y - 3)^2 = z^2$$

It is fairly obvious that this equation has no solution because "no perfect square can be the sum of two odd squares" which is a rule based on the Pythagorean triple rules and properties (also see § 13).[10]

6.2.3 Pell's Equation

We should also remember Pell's equation when we deal with quadratic polynomial equations in two variables with no mixed-variables terms. As before, we should consider manipulating or/and transforming the equation (if necessary) to put it in a standard Pell equation form (if possible), e.g. by scaling the equation and completing squares. In fact, such manipulations and transformations are generally useful (regardless of the possibility of putting the equation in Pell equation form or not) because they usually simplify the expressions in the equation and reduce the number of its terms (as well as organizing the equation in general). All these benefits should help to identify and recognize potential patterns that can indicate the solutions or the method of solution (whether by Pell techniques or by something else).

For example, the following Diophantine equation:

$$x^2 - 2y^2 = 1 \qquad\qquad (\text{or } x^2 - 1 = 2y^2)$$

is already in a standard Pell equation form (or almost) and hence its solution can be obtained readily by the standard techniques of solving Pell's equations. So, virtually no

[10] One of the Pythagorean triple rules is: if (a, b, c) is a primitive Pythagorean triple then a and b have opposite parity (noting that multiplication by neither odd factor nor even factor can change the parity in a way that makes both a and b odd).

effort is required to solve this equation.

On the other hand, the following Diophantine equation:

$$4x^2 - 6y^2 + 12x + 108y - 478 = 0$$

is not in a standard Pell equation form but it can be manipulated and transformed to a standard Pell equation form with some minimal effort, that is:

$$X^2 - 6Y^2 = 1$$

where $X = 2x + 3$ and $Y = y - 9$. Now, in this form it can be easily solved as a Pell equation and hence the solutions of the original equation can be obtained by the reverse transformations, i.e. $x = (X - 3)/2$ and $y = Y + 9$.

However, it is important to remember the following notes and recommendations when dealing with polynomial Diophantine equations which are supposed to be solved (directly or indirectly) by the Pell equation methods:

1. We should consider in this regard the generalized form of Pell's equation (i.e. not only its basic form).[11]
2. When tackling a Pell equation (whether transformed or not) it is recommended to search for the fundamental solution by inspection before trying sophisticated processes and techniques (like continued fractions technique). For example, a few-seconds inspection to the Pell equation:

$$x^2 - 5y^2 = 1$$

   should lead to the fundamental solution $(x_1, y_1) = (9, 4)$ and hence to all other solutions using the well-known formulations and instructions (related to Pell equation solution). Also basic computational effort (e.g. through the use of a spreadsheet or a simple code) can be more economic in the search for the fundamental solution.
3. As soon as we solve a Pell equation obtained by manipulations and transformations we can try obtaining the solutions of the original equation by the reverse transformations. However, since we accept only the integer solutions to the original equation, the existence of solution to the transformed Pell equation does not guarantee the existence of solution to the original equation because the reverse transformations may not produce integer solutions to the original equation. For example, the following Diophantine equations:

$$\begin{aligned} 4x^2 - 6y^2 + 12x + 108y - 478 &= 0 \\ 9x^2 - 325y^2 - 42x - 130y + 35 &= 0 \end{aligned}$$

[11] Pell's equation is generalized to the form $x^2 - dy^2 = c$ where $0 \neq 1 \neq c \in \mathbb{Z}$.

$$10x^2 + 2x - 8y^2 = 0$$

can be easily transformed to the following standard Pell equation form:

$$\begin{aligned} X^2 - 6Y^2 &= 1 & \quad (X = 2x + 3,\ Y = y - 9) \\ X^2 - 13Y^2 &= 1 & \quad (X = 3x - 7,\ Y = 5y + 1) \\ X^2 - 80y^2 &= 1 & \quad (X = 10x + 1) \end{aligned}$$

All these Pell equations have solutions. However:

• The first original equation accepts all the solutions obtained by the reverse transformations (because the reverse transformations produce integer solutions to the original equation in all cases).

• The second original equation does not accept any of the solutions obtained by the reverse transformations (because the reverse transformations do not produce integer solutions to the original equation in any case).

• The third original equation accepts only some of the solutions obtained by the reverse transformation (because the reverse transformation produces integer solutions to the original equation only in some cases).

4. The usefulness of Pell equation (and the possibility of its exploitation) in solving Diophantine equations is not limited to solving the given equation directly (with or without manipulations and transformations) but it extends beyond this. For example, the following Diophantine equation:

$$x^2 + x - 2y^2 = 0$$

can be treated as a (one-variable) quadratic in x and hence it has a solution if its discriminant Δ is a perfect square, i.e.

$$\Delta = 1^2 - 4(-2y^2) = k^2 \qquad \rightarrow \qquad k^2 - 8y^2 = 1 \qquad\qquad (k \in \mathbb{N})$$

As the latter is a Pell equation it has solutions (and actually infinitely many solutions), and hence the original Diophantine equation must also have solutions (and actually infinitely many solutions).

6.2.4 Modular Reduction

A well-known approach for solving Diophantine equations in general (especially those of polynomial types) is the reduction of equation in an appropriate modulo (and possibly in more than one modulo) which can reduce the number of variables (or/and show certain features or patterns) and hence simplify the search for solution. The technique of modular reduction (which is widely used for solving Diophantine equations whether of polynomial types or other types) will be investigated further later on (see § 7). So, here we only give some simple examples for the use of this technique for solving polynomial Diophantine equations.

For example, we can easily *solve* the following polynomial Diophantine equation (i.e. find out that it has no solution in $\mathbb{Z}$):

$$15x^2 - 35y^3 = 10 \tag{1}$$

by reducing it in modulo 7 to get: $x^2 \stackrel{7}{=} 3$ which obviously has no solution (since 3 is not a quadratic residue of 7) and hence we can easily conclude that the original Diophantine equation has no solution (see § 7.1).

Similarly, the Diophantine equation:

$$x^2 - 3y - 2z = 0 \tag{2}$$

can be reduced in modulo 2 to get: $x^2 - y \stackrel{2}{=} 0$ which is much easier to analyze and solve. Further investigation (based on considering the parity of x and y) should lead to the solution of $x^2 - y \stackrel{2}{=} 0$ and hence to the solution of $x^2 - 3y - 2z = 0$.[12]

More examples about solving non-linear polynomial Diophantine equations (as well as other types of Diophantine equations) by the use of modular arithmetic reduction technique will be given in the future.

In this regard, we should remember the following recommendations and guidelines (or "rules") which we can gather from personal experience (as well as from the literature) about the use of modular reduction in solving polynomial Diophantine equations:

1. Always try to use small reduction moduli since they are easier in management and analysis. In general, when using modular reduction consider starting your investigation and analysis with the smallest moduli first (i.e. use moduli in increasing order).
2. In general, modular reduction can lead conclusively to the non-existence of solution but not to the existence of solution (let alone finding the specific solution) because having a modular solution is more general than having a solution to the corresponding Diophantine equation (i.e. having a modular solution is a necessary but not sufficient condition for having a solution to the corresponding Diophantine equation). However, modular reduction can lead to finding the solution (through initiating or outlining a logical/mathematical argument or by showing a certain pattern for instance). So, it is useful to try pursuing the consequences of modular reduction analysis even when

[12] The solutions are (where $k, s \in \mathbb{Z}$): $(x, y, z) = (2k, 2s, 2k^2 - 3s)$ and $(2k+1, 2s+1, 2k^2 + 2k - 3s - 1)$.

modular reduction lead to having modular solution. Such further analysis can lead to the conclusion of having a solution to the Diophantine equation (and even to finding the solution specifically).

3. Consider using more than one reduction modulo (i.e. in more than one modular reduction operation) associated with comparison and analysis of the results of the different reduction moduli. Such comparison and analysis can lead to producing a logical/mathematical argument that leads to the solution of the problem in hand.

6.2.5 Factorization Analysis

Another well-known approach for solving Diophantine equations in general (including those of polynomial types) is factorization or/and comparison of factors. In fact, this method is diverse and versatile and hence in the following we outline a few common types of this method with some simple and illuminating examples:

1. A common type of this method is to put the given Diophantine equation in such a form where a factorizable algebraic expression becomes equal to a (factorizable) specific integer and hence a comparison between the factors on the two sides of the equation produces systems of simultaneous equations whose solutions produce all the solutions of the original Diophantine equation. For example, the following Diophantine equation:

$$x^2 + y^2 + 18xy - x^2y^2 - 81 = 0$$

can be put in the following form:

$$(x + y)^2 - (xy - 8)^2 = 17$$

Now, both sides are factorizable, i.e.

$$\begin{aligned} (x + y)^2 - (xy - 8)^2 &= (x + y - xy + 8)(x + y + xy - 8) \\ \text{and} \qquad 17 &= 1 \times 17 = (-1) \times (-17) \end{aligned}$$

Now, if we consider all the possible combinations of these factorization possibilities of the two sides in both orders then we get four systems of simultaneous equations whose solutions (if exist) produce all the solutions of the given Diophantine equation.

Other examples of this type are the equations:

$5x + xy - 2y = 0 \qquad x^2 - y^2 + 8y - 28 = 0 \qquad x^2 - y^2 - 12x - 3y + 1 = 0$

which can be put in the following forms:

$(2-x)(y+5) = 10 \qquad (x-y+4)(x+y-4) = 12 \qquad (2x-2y-15)(2x+2y-9) = 131$

and hence they can be analyzed and solved in a similar manner.

2. Another type of this method is to manipulate the equation to make a variable equal to a ratio (or quotient) whose numerator and denominator can be compared and analyzed to deduce the possible solutions of the original Diophantine equation. For example, the following Diophantine equation:

$$x^3y - 125x + 125 = 0$$

can be put in the following form:

$$y = \frac{125(x-1)}{x^3}$$

Now, it is obvious that $(x-1)$ and x^3 are coprime and hence x^3 must divide 125. This means that x can only be ±1 and ±5 and these values of x should produce all the possible values of y and hence we get all the possible solutions of the original Diophantine equation.

3. An example of a type similar to the previous two types is the equation:

$$x^6y + xy^6 - 256 = 0$$

which can be put in the following two forms:

$$x(x^5y + y^6) = 256 \qquad \text{and} \qquad y(x^6 + xy^5) = 256$$

These forms should indicate that x and y must be divisors of 256. So, by testing all the possibilities of x being equal to one of the 18 divisors of 256 and y being equal to one of these 18 divisors we find that only one possibility (i.e. $x = y = 2$) satisfies the given Diophantine equation and hence we obtain the required solution.

Also see § 8.

6.2.6 One-Variable Quadratic Approach

In some cases it is possible to solve a quadratic Diophantine equation in two variables (possibly with a mixed-variable term) by treating it as a quadratic equation in one variable whose discriminant Δ can be inspected and analyzed to deduce the solution of the given Diophantine equation.

For example, the following Diophantine equation:

$$5x^2 - 8xy + 11y^2 - 1175 = 0$$

can be treated as a quadratic in x and hence we form and inspect its discriminant Δ, that is:

$$\Delta = 64y^2 - 20(11y^2 - 1175) = -156y^2 + 23500 > 0 \qquad \to \qquad y < \sqrt{\frac{23500}{156}} \simeq 12.27$$

Now, if we consider first only the positive values of y then the only possibilities we have are $y = 1, 2, \ldots, 12$. On testing these possible values on the original equation we get only three solutions, i.e. $(x, y) = (-10, 5)$, $(18, 5)$ and $(2, 11)$. Now, if we note that the equation is indifferent to change of sign of both variables (noting the mixed-variable term) then we can conclude that we must have three other solutions, i.e. $(x, y) = (10, -5)$, $(-18, -5)$ and $(-2, -11)$. So, by this way we obtain all the possible solutions of this equation in $\mathbb{Z}$.

A second example is the following Diophantine equation:

$$x^2 - xy + 6x - y + 2 = 0$$

which can be put in the following form:

$$x^2 + (6 - y)x + (2 - y) = 0$$

and hence it can be treated as a one-variable quadratic in x whose discriminant must be a perfect square (say k^2 where $k \in \mathbb{Z}$), that is:

$$\Delta \equiv (6 - y)^2 - 4(2 - y) = k^2 \qquad \to \qquad (k - y + 4)(k + y - 4) = 12$$

On inspecting and analyzing this discriminant (using factorization and comparison as indicated in the last equation; see § 6.2.5), all the solutions of the given Diophantine equation can be easily obtained.

A third example is the following Diophantine equation:

$$x^2 + xy + 2y + 2 = 0$$

which can be treated as a one-variable quadratic equation in x whose discriminant is:

$$\Delta \equiv y^2 - 8y - 8 = k^2 \qquad \to \qquad (y - 4 - k)(y - 4 + k) = 24$$

On inspecting and analyzing the last equation (as in the previous example) we get all the solutions.

Also see the example in point 4 of § 6.2.3.

6.2.7 Separation of Variables with Divisibility Analysis

Some polynomial Diophantine equations in two variables can be manipulated such that one variable is separated and equated to a rational fraction involving functions of only the other variable. For example, the equation:

$$3x + 5xy - 6y - 5 = 0$$

can be manipulated to become:

$$y = \frac{5 - 3x}{5x - 6}$$

Now, if we apply a divisibility analysis on the RHS of this equation then we can conclude that the denominator must be ± 1 or ± 7. Further algebraic analysis will lead to the conclusion that the given Diophantine equation has only one acceptable solution which is $(x, y) = (1, -2)$.

This method similarly applies to the equation:

$$x^2 - 9x - 4y - xy + 13 = 0$$

which can be manipulated to become:

$$y = \frac{x^2 - 9x + 13}{x + 4} = x - 13 + \frac{65}{x + 4}$$

A divisibility analysis on the RHS will lead to the conclusion that $x + 4$ must be ± 1, ± 5, ± 13, ± 65. With further algebraic analysis all the eight solutions of the given Diophantine equation can be easily found.

6.2.8 Other Recommendations and Considerations

Other recommendations and considerations that should be remembered when dealing with non-linear polynomial Diophantine equations include (among many other things):

1. Considering upper and lower bounds on the potential solutions (see § 11).
2. Recalling and employing basic rules and principles (see § 13).
3. Other general recommendations and guidelines that will be investigated or outlined later on (see for instance § 14).

6.3 Exponential Equations

Diophantine exponential problems (whether individual equations or systems of equations) come in many different shapes and forms. Except in very simple cases, solving these problems generally poses a serious challenge. In fact, they are usually more difficult to solve than the more common type of Diophantine polynomial problems. Even searching for their solutions computationally can pose a challenge (in comparison for instance to such search for the solutions of polynomial problems) due to the rapid and explosive rise in the magnitude of the inspected values which imposes serious limitations on the range of inspected values unless we have access to exceptional computational facilities and techniques.

In the following sub-subsections we will try to discuss and investigate some issues (such as recommendations and techniques) which are useful to know and remember when tackling simple types of Diophantine exponential problems. The last sub-subsection of this subsection is dedicated to some general recommendations and guidelines about the use of the technique of modular reduction and analysis in solving Diophantine exponential equations (and equations involving exponentials) noting that modular reduction and analysis is the most common approach for tackling and solving Diophantine exponential equations (and equations involving exponentials) and hence it enjoys a special importance that requires special attention.

However, before we go through these details we would like to outline a general strategy for tackling and solving Diophantine exponential equations. In brief, the solutions of Diophantine exponential equations (and equations involving exponentials) are usually at the very low values of their variables (when solutions exist). Hence, the best strategy for solving these equations is to inspect these equations computationally (see § 5) for solutions within a certain range of their variables that includes low values of their variables (e.g. between 0 and 15) and hence we have three main cases and scenarios (which determine our strategy and approach):

- No solution found: in this case we look for an argument (e.g. based on parity violation or modular contradiction) to prove that the given equation has no solution.
- A few solutions found (usually in the neighborhood of zero): in this case we look for an argument showing that there are no other solutions (i.e. at high values of their variables).
- Considerable number of solutions found: in this case we look for a pattern within the found solutions on which we can build an argument for the type of solutions (where such an argument should establish whether there are other solutions or not).

6.3.1 Magnitude Consideration

The very basic (and possibly trivial) Diophantine exponential equations can be solved by just inspecting the limitation on the magnitude of the two sides of the equation. For example, the following Diophantine exponential equations (where $x, y \in \mathbb{N}^0$):

$$2^x + 3^y = 1 \qquad\qquad 4^x + 9^y = 2 \qquad\qquad 5^x + 7^y = 40369232$$

can be easily "solved" by noting that the first equation has no solution because the LHS cannot be less than 2, and the second equation has only the trivial solution $x = y = 0$ because otherwise the LHS will be greater than 2, while the third equation (assuming it is solvable) must have solution(s) only for very low values of its variables and hence the solution(s) can be obtained computationally (knowing for sure, by this logical argument, that there is no other solution within the high values of its variables).[13]

6.3.2 Modular Reduction

The method of modular reduction is as common in solving exponential Diophantine equations as in solving polynomial Diophantine equations (see § 6.2.4 and § 7). As indicated earlier (and later), one of the common purposes of modular reduction is to simplify the equation (e.g. by removing some variables) or/and showing and revealing some "hidden" patterns and clues that can help in solving the Diophantine equation (noting that the employed modular reduction may not introduce any "visible" change on the original Diophantine equation apart from converting it from an ordinary equation to a congruence equation). The logical and mathematical foundations for the use of modular reduction in solving Diophantine equations will be outlined later on (see § 7.1) and hence the unfamiliar reader should refer to that subsection.

For example, we can easily *solve* the following exponential Diophantine equation (i.e. find out that it has no solution in $\mathbb{N}^0$):

$$4^x - 12^y = a \qquad\qquad (a = 1 \text{ or } a = 6 \text{ or } a = 7) \tag{3}$$

by reducing it in modulo 13 to get: $4^x - 12^y \overset{13}{=} a$ which has no solution for $a = 1, 6, 7$ (since $1, 6, 7$ are not modular values of $4^x - 12^y$ for any combination of $x, y \in \mathbb{N}^0$) and hence we can easily conclude that the original Diophantine equation has no solution (see § 7.1).

Similarly, the exponential Diophantine equation:

$$4^x - 3^y = 1 \qquad\qquad (x, y \in \mathbb{N}^0)$$

can be solved by noting that for $x > 1$ we have:

$$4^x - 3^y = 2^{2x} - 3^y = (2^3 \times 2^{2x-3}) - 3^y = (8 \times 2^{2x-3}) - 3^y \overset{8}{=} -3^y \overset{8}{\neq} 1$$

and hence it has no solution. So, if it has any solution then x must be 0 or 1. A simple inspection will then reveal that the only possible solution of this Diophantine equation is $(x, y) = (1, 1)$. The same method and argument apply to the exponential Diophantine

[13] Actually, the third equation has only one solution which is $x = 6$ and $y = 9$.

equation:

$$4^x - 3^y = 3 \qquad\qquad (x, y \in \mathbb{N}^0)$$

where we can prove that the only possible solution of this equation is $(x, y) = (1, 0)$.

There are many other examples (some of which will be given later) for the use of modular reduction as a tool for solving exponential Diophantine equations.

It is important to note that in many cases modular analysis may require using more than one modulo for reduction and analysis. For example, the equation:

$$5^x - 3^y = 2 \tag{4}$$

can be solved by reducing it in modulo 9 where the solution $(x, y) = (1, 1)$ is found for the case of $3^y \overset{9}{=} 3$ (i.e. $5^x - 3^1 = 2$). Further analysis in modulo 28 is then applied to show that there is no solution in the other case (i.e. $3^y \overset{9}{=} 0$ or $x \overset{6}{=} 5$).

Another example is the equation:

$$3^x + 5^y = z^2 \tag{5}$$

where its analysis in modulo 3 and modulo 9 leads to obtaining its two solutions, i.e. $(x, y, z) = (1, 0, \pm 2)$.

Also see the examples of § 6.3.3.

6.3.3 Parity and Modular Analysis

In many cases, Diophantine exponential equations can be solved by a combination of parity and modular analysis (noting that parity analysis is a form of modular analysis; see § 2.5). For example, the equation:

$$2^x - 3^y = 1$$

can be solved by considering the parity of x where the equation is reduced in modulo 3 for odd x to get the solution $(x, y) = (1, 0)$, and analyzed in modulo 8 for even x to get the solution $(x, y) = (2, 1)$.

A similar method applies to the equation:

$$3^x - 2^y = 1$$

by considering the parity of x where the equation is reduced in modulo 8 for odd x to get the solution $(x, y) = (1, 1)$. The equation is then analyzed further for even x where it is factorized as $(3^k - 1)(3^k + 1) = 2^y$ and analyzed to infer the power of 2 to which these factors correspond and this analysis leads to the only solution in this case which is

$(x, y) = (2, 3)$.

6.3.4 Recommendations about Modular Analysis

We would like to draw the attention in this sub-subsection to the following recommendations and guidelines which we can gather from our personal experience (as well as from the literature) about the use of the technique of modular reduction in solving exponential Diophantine equations:[14]

1. Try to find and use a reduction modulo that is as small as possible so that you deal with small or modest residue system that is easy to manage and deal with. A systematic inspection to candidate reduction moduli in their increasing magnitude should lead to the identification of the smallest modulo that is appropriate for that purpose (if it exists).
2. Try to choose (when you have a choice) a reduction modulo k so that the base of interest n for that reduction modulo has a low integer order, i.e. $O_k n$.[15] For instance, if the base of interest is 7 (e.g. we are dealing with 7^x) then $O_{10}7 = 4$ while $O_{13}7 = 12$ and hence if we have a choice then 10 is a better reduction modulo. The reason for this is that low integer order means fewer cases to deal with and easier investigation and management.
3. Try to choose (when you have a choice) a reduction modulo that is prime (not composite) or at least a power of prime. This should reduce the complications due to the obvious advantages of dealing with prime moduli (although this can be in conflict with the previous recommendation).
4. Consider using more than one reduction modulo (i.e. in more than one modular reduction operation) associated with comparison and analysis of the results of the different reduction moduli. Such comparison and analysis can lead to producing a logical/mathematical argument that leads to the solution of the problem in hand. Some examples for using more than one modulo were given earlier (see for instance the analysis of Eqs. 4 and 5).
5. There is no guarantee that the problem in hand can be tackled by modular reduction in an appropriate way (commensurate with the nature and size of the problem) and hence

[14] In fact, most of these recommendations and guidelines also apply to other types of Diophantine equations.

[15] We note that "the base of interest" is more general than dealing with only one powered base (i.e. when modular reduction eliminates the powers of other bases) and dealing with more than one powered base (i.e. when more than one powered base remain after modular reduction).

be prepared for possible failure. Accordingly, be prepared to give up after reasonable time and effort (i.e. don't be bogged down!).[16]

6.4 Mixed Polynomial-Exponential Equations

There is no single standard method or technique for solving this type of Diophantine equations. However, the methods and techniques used for solving polynomial and exponential equations (see § 6.1, 6.2 and 6.3) are generally used to solve this type of Diophantine equations (since the equation is a mix of these types of Diophantine equations). It is important to note in this context the special importance of using modular reduction which is seemingly the most common technique in tackling this type of problems (although it is usually associated with other techniques and tricks and further modular and non-modular analysis). This should be understandable noting the ability of modular reduction to eliminate one type of the mix (i.e. the polynomial type and the exponential type) if such elimination is required as well as its exceptional ability to simplify the equation in general (with and without elimination of one type) and expose it to the versatile (and relatively simple) tools and rules of modular arithmetic. We should also mention in this regard that the use of modular reduction is common in tackling both polynomial and exponential types of Diophantine equations (see § 6.2.4 and 6.3.2) and hence it should also be common in tackling mixed polynomial-exponential Diophantine equations.

In the following sub-subsections we will present a sample of this type of Diophantine equations and the methods used in their solutions. This should give an idea about the methods used in solving this type of equations.

6.4.1 Simple Inspection

Some mixed polynomial-exponential Diophantine equations are so trivial that they can be "solved" by just simple inspection. For example, if we write the equation:

$$7^x - 8^y - z = 0 \qquad\qquad (x, y \in \mathbb{N}^0,\ z \in \mathbb{Z})$$

as $z = 7^x - 8^y$ then it is obvious that there is no restriction on z (other than being integer) and hence we are free to assign any value (within the domain) to its two exponential variables which they fix (as soon as they are assigned) the value of z. So, the solutions of this equation are obviously $(x, y, z) = (k, s, 7^k - 8^s)$ where $k, s \in \mathbb{N}^0$.

[16] This recommendation (in our view) is especially important when dealing with exponential equations because the search for an appropriate reduction modulo in this case is usually demanding and time consuming and hence it can waste considerable amount of time if we keep trying and trying.

6.4.2 Inspection with Modular Analysis

For example, the equation:

$$2^x + 3^y = z^2 \qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z}) \tag{6}$$

can be inspected (aided by modular analysis) for low values of x where six solutions can be found, i.e. $(x, y, z) = (0, 1, \pm 2)$, $(3, 0, \pm 3)$ and $(4, 2, \pm 5)$. On applying further modular analysis (in modulo 8) we can conclude that this Diophantine equation has no solution other than these six. This method also applies to the equation:

$$2^x - 3^y = z^2 \qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z}) \tag{7}$$

whose five solutions at low values of x $\big[$namely $(x, y, z) = (0, 0, 0)$, $(1, 0, \pm 1)$ and $(2, 1, \pm 1)\big]$ can be easily found by inspection while modular analysis (in modulo 8) can be used to prove that it has no other solutions.

Another example of this method is the equation:

$$3^x - 4^y = z^2 \qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z})$$

which can be inspected for $x = 0$ to find the trivial solution $(x, y, z) = (0, 0, 0)$. Modular analysis (in modulo 3) can then easily reveal that this equation has no solution for $x > 0$ and hence the equation has only the trivial solution. This method also applies to the equation:

$$4^y - 3^x = z^2 \qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z})$$

whose three solutions at low values of y $\big[$namely $(x, y, z) = (0, 0, 0)$, $(1, 1, \pm 1)\big]$ can be easily found by inspection while modular analysis (in modulo 8) will reveal that it has no other solutions.

6.4.3 Comparison to a Similar Equation

For example, the equation:

$$4^x + 3^y = z^2 \qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z})$$

can be solved by writing it as $2^X + 3^y = z^2$ (where $X = 2x$) and hence comparing it to Eq. 6. As we see, this equation is the same as Eq. 6 (with X replacing x). Now, if we note that $X = 2x$ then we can conclude (by using the solutions of Eq. 6) that the only solutions to the given equation are $(x, y, z) = (0, 1, \pm 2)$ and $(2, 2, \pm 5)$.

6.4.4 Modular Reduction with Further Analysis

For example, the equation:

$$5^x - 6y + 21 = 0 \qquad\qquad (x \in \mathbb{N}^0, y \in \mathbb{Z})$$

can be solved by modular reduction (mod 6) to get $5^x + 21 \stackrel{6}{=} 0$, i.e. $(-1)^x + 3 \stackrel{6}{=} 0$. This congruence equation has no solution (because the left hand side is either 2 or 4) and hence the given Diophantine equation has no solution.

Similarly, the equation:

$$3^x + 5^y - 4z - 2 = 0 \qquad\qquad (x, y \in \mathbb{N}^0, z \in \mathbb{Z})$$

can be solved by modular reduction (mod 4) to get $3^x+5^y-2 \stackrel{4}{=} 0$, i.e. $(-1)^x+(1)^y-2 \stackrel{4}{=} 0$. The solution of this equation is obviously $x = 2k$ and $y = s$ ($k, s \in \mathbb{N}^0$). On solving the given equation for z we get $z = (3^{2k} + 5^s - 2)/4$ and hence the solutions of the given equation are all triples of the following form: $(x, y, z) = \left(2k,\ s,\ \frac{3^{2k}+5^s-2}{4}\right)$. It is worth noting that $(3^{2k} + 5^s - 2)/4$ is integer for all $k, s \in \mathbb{N}^0$ (since $3^{2k} + 5^s - 2$ is zero in modulo 4) and hence this solution applies to all $k, s \in \mathbb{N}^0$.

6.4.5 Modular Reduction with Substitution

For example, the equation:

$$5x + 4^y - 11 = 0 \qquad\qquad (x \in \mathbb{Z}, y \in \mathbb{N}^0)$$

can be solved by modular reduction (mod 5) to get $4^y - 11 \stackrel{5}{=} 0$, i.e. $(-1)^y - 1 \stackrel{5}{=} 0$. The solution of this congruence equation is all even $y \ge 0$, i.e. $y = 2k$ ($k \in \mathbb{N}^0$). On substituting this into the given Diophantine equation and solving the resulting equation for x we get $x = (11 - 4^{2k})/5$ and hence the solutions of the given equation are all pairs of the following form: $(x, y) = \left(\frac{11-4^{2k}}{5},\ 2k\right)$ where $k \in \mathbb{N}^0$. It is worth noting that $(11-4^{2k})/5$ is integer for all values of k because for $k = 0$ it is equal to 2, while for $k > 0$ the numerator $(11 - 4^{2k})$ ends in 5 and hence it is divisible by 5.

6.4.6 Modular Reduction with Parity Analysis

For example, the equation:

$$5^x - 11x + 3y + 1 = 0 \qquad\qquad (x \in \mathbb{N}^0, y \in \mathbb{Z})$$

can be solved by modular reduction (mod 3) to get $(-1)^x + x + 1 \stackrel{3}{=} 0$ whose solutions (which can be inferred by parity analysis of x aided by modular inspection) are $x = 3+6k$

and $x = 4 + 6k$ where $k \in \mathbb{N}^0$.[17] On substituting these expressions of x in the original Diophantine equation and solving for y we get the required solutions, i.e.

$$(x, y) = \left(3 + 6k, \frac{-5^{3+6k} + 32 + 66k}{3}\right) \quad \text{and} \quad (x, y) = \left(4 + 6k, \frac{-5^{4+6k} + 43 + 66k}{3}\right)$$

It is straightforward to show that $(-5^{3+6k} + 32 + 66k)/3$ and $(-5^{4+6k} + 43 + 66k)/3$ are always integers and hence these solutions are valid for all values of $k \in \mathbb{N}^0$.

6.4.7 Classification with Parity Analysis

For example, the equation:

$$7^x - 8^y - 2z = 0 \qquad\qquad (x, y \in \mathbb{N}^0,\ z \in \mathbb{Z})$$

can be solved by classifying it according to the value of its exponential variables associated with parity analysis. More specifically, the case $x = 0$ and $y > 0$ as well as the case $x > 0$ and $y > 0$ have no solution due to parity violation. So, all we need to consider is the case $x = y = 0$ which leads to the trivial solution $(x, y, z) = (0, 0, 0)$ and the case $x > 0$ and $y = 0$ which leads to the obvious solution $(x, y, z) = \left(k, 0, \frac{7^k-1}{2}\right)$ where $k \in \mathbb{N}$.

6.5 Equations Involving Roots

Equations involving roots may not be classified technically as Diophantine equations although this will not prevent us from including them in our investigation due to the obvious merit and justification of this inclusion and their undeniable qualification to be treated as such. There are a variety of methods (or techniques or tricks or ...) for tackling Diophantine equations involving roots. In the following sub-subsections we present some of the most common of these methods with illuminating and illustrating examples.

6.5.1 Simple Inspection

Some of the Diophantine equations involving roots are so trivial that they can be "solved" by just simple inspection. For example, the equation:

$$\sqrt{x} - y = 379$$

can be solved by noting that x must be a perfect square (i.e. $x = s^2$ where $s \in \mathbb{Z}$) and hence $y = \sqrt{x} - 379 = |s| - 379$ (where $|s|$ is the absolute value of s). So, the solutions

[17] We note that if we reduce the equation (mod 3) to the form $5^x + x + 1 \stackrel{3}{=} 0$ then we can use a different method of analysis (based on inspection and induction).

are $(x,y) = (s^2, |s| - 379)$ where $s \in \mathbb{Z}$.

Another example is the equation:

$$5^x - 7^y - 2\sqrt{z} = 0 \qquad\qquad (x,y,z \in \mathbb{N}^0)$$

which can be written as $\sqrt{z} = (5^x - 7^y)/2$ and hence $z = \left([5^x - 7^y]/2\right)^2$ where $(5^x - 7^y) \geq 0$. So, the solutions of the given equation are all triples of the following form: $(x,y,z) = \left(k,\ s,\ \left[\frac{5^k - 7^s}{2}\right]^2\right)$ where $k,s \in \mathbb{N}^0$ and $(5^k - 7^s) \geq 0$. It is worth noting that z is always integer because $(5^k - 7^s)$ is even for all $k,s \in \mathbb{N}^0$.

6.5.2 Enumeration

For example, the equation:

$$\sqrt{x} + \sqrt{y} = 9$$

can be solved by noting that there are only 10 pairs of $\sqrt{x}$ and $\sqrt{y}$ that can add up to 9, i.e. $(\sqrt{x}, \sqrt{y}) = (0,9), (1,8) \ldots (9,0)$. So, if we square the numbers in each pair then we get all the possible solutions. So, the 10 solutions are: $(x,y) = (0,81),\ (1,64), \ldots (81,0)$.

This similarly applies to the equation:

$$\sqrt{x} + \sqrt{y} = \sqrt{363}$$

which can be solved by noting that $\sqrt{363} = 11\sqrt{3}$ and hence we have only 12 pairs of $\sqrt{x}$ and $\sqrt{y}$ that can add up to $\sqrt{363}$, i.e. $(\sqrt{x}, \sqrt{y}) = (0\sqrt{3}, 11\sqrt{3}), (1\sqrt{3}, 10\sqrt{3}), \ldots, (11\sqrt{3}, 0\sqrt{3})$. So, if we square the numbers in each pair then we get all the 12 possible solutions, i.e. $(x,y) = (0,363),\ (3,300), \ldots (363,0)$.

6.5.3 Linearization

For example, the equation:

$$6x + 10\sqrt{y} - 19z = 0$$

can be solved by letting $Y = \sqrt{y}$ and hence we get $6x + 10Y - 19z = 0$ whose solution is $(x,Y,z) = (11s - 19k,\ s,\ 4s - 6k)$ where $s,k \in \mathbb{Z}$. However, $\sqrt{y}$ must be an integer and hence we must have $y = t^2$ $(t \in \mathbb{Z})$. Therefore, $Y = \sqrt{y} = \sqrt{t^2} = |t| = s$. Thus, the solution of the original equation is $(x,y,z) = (11|t| - 19k,\ t^2,\ 4|t| - 6k)$ where $t,k \in \mathbb{Z}$.

Another example of linearization is the equation:

$$21x + 35\sqrt{y} - 12\sqrt{z} = 41$$

which can be solved by letting $Y = \sqrt{y}$ and $Z = \sqrt{z}$ and hence we get $21x + 35Y - 12Z = 41$ whose solution is $(x, Y, Z) = (2 - 5s - 4k,\ 1 + 3s,\ 3 - 7k)$ where $s, k \in \mathbb{Z}$. Now, $Y = \sqrt{y} \geq 0$ and hence s must be ≥ 0 (i.e. $s \in \mathbb{N}^0$). similarly, $Z = \sqrt{z} \geq 0$ and hence k must be ≤ 0 (i.e. $\mathbb{Z} \ni k \leq 0$). Therefore, the solution of the original equation is $(x, y, z) = (2 - 5s - 4k,\ [1 + 3s]^2,\ [3 - 7k]^2)$ where $s \in \mathbb{N}^0$ and $\mathbb{Z} \ni k \leq 0$.

The equation:

$$\sqrt{x+1} - \sqrt{y+5} = 1$$

can also be solved by linearization by letting $X = (x + 1)$ and $Y = (y + 5)$ and hence we have $\sqrt{X} - \sqrt{Y} = 1$. Now, if $X = k^2$ $(k \in \mathbb{Z})$ then $\sqrt{Y} = \sqrt{X} - 1 = |k| - 1$, i.e. $Y = (|k| - 1)^2$. However, since $\sqrt{X} - \sqrt{Y} = 1$ then we must have $X > Y$ which implies $|k| > (1/2)$, i.e. $k \in \mathbb{Z}$ and $k \neq 0$. Hence, $X = (x + 1) = k^2$ and $Y = (y + 5) = (|k| - 1)^2$ where $k \in \mathbb{Z}$ and $k \neq 0$. So, the solutions of the given equation are all pairs of the following form: $(x, y) = (k^2 - 1,\ \{|k| - 1\}^2 - 5)$ where $k \in \mathbb{Z}$ and $k \neq 0$.

6.6 Equations Involving Fractions

Again, equations involving fractions may not be classified technically as Diophantine equations although this will not affect our decision to include them in our investigation (justified at least by practical considerations even if they are not Diophantine equations from a theoretical viewpoint due to the limitation of the definition of Diophantine equations). Anyway, there are various methods and techniques for tackling Diophantine equations involving fractions (depending for instance on their types and number of variables). In the following sub-subsections we will outline some of the most common of these methods presented and demonstrated within illuminating examples.

6.6.1 Magnitude Analysis

For example, the equations:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 4 \qquad \text{and} \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \qquad (x, y, z \in \mathbb{Z},\ xyz \neq 0)$$

can be easily solved by magnitude analysis where the first equation has no solution because its LHS cannot be greater than 3 (i.e. when $x = y = z = 1$), while the second equation has only one solution (i.e. $x = y = z = 1$) because if any one of the variables is not equal to 1 then the LHS will be less than 3.

6.6.2 Sign and Magnitude Analysis

For example, the equation:

$$\frac{x}{y} + \frac{y}{x} = 1 \qquad\qquad (x, y \in \mathbb{Z},\ xy \neq 0)$$

can be solved by noting that if this equation has a solution then x and y must have the same sign (because otherwise the sum will be negative) and hence (whether $x \geq y$ or $x < y$) the LHS must be greater than 1, i.e. the equation has no solution.

Many similar equations can be analyzed and solved by this method, e.g. the equation:

$$\frac{x}{y} + xy = 1 \qquad\qquad (x, y \in \mathbb{Z},\ y \neq 0)$$

has no solution because x and y must have the same sign (otherwise the sum will be negative) and hence (whether $x \geq y$ or $x < y$) the LHS must be greater than 1 (noting that $x = 0$ is not a possibility).

This also applies to the equations:

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{3} \qquad\qquad \frac{1}{x} + \frac{2}{y} = \frac{3}{4} \qquad\qquad (x, y \in \mathbb{Z},\ xy \neq 0)$$

which can be solved by this method (where the details can be found in [3]).

6.6.3 Divisibility Analysis

Some equations can be manipulated in one (or more) form that facilitates simple divisibility analysis which leads to the solutions. For example, the equation:

$$\frac{1}{x} + \frac{1}{y} = z \qquad\qquad (x, y, z \in \mathbb{Z},\ xy \neq 0)$$

can be manipulated into the following two forms:

$$1 + \frac{x}{y} = xz \qquad\qquad \frac{y}{x} + 1 = yz$$

which imply $y = \pm x$ and this (with extra basic analysis) will lead to the required solutions.

6.6.4 Separation of Variables with Divisibility Analysis

For example, the equation:

$$\frac{14}{x} + \frac{y}{19} = 25 \qquad\qquad (x, y \in \mathbb{Z},\ x \neq 0) \tag{8}$$

can be manipulated to become:

$$y = 475 - \frac{266}{x}$$

Now, by a basic divisibility analysis (i.e. x must be a divisor of 266) we can easily obtain all the (sixteen) solutions of the original Diophantine equation.

This method also applies to the equation:

$$\frac{20}{x} + \frac{33}{y} = 2 \qquad\qquad (x, y \in \mathbb{Z},\ xy \neq 0) \tag{9}$$

which can be manipulated to become:

$$x = \frac{20y}{2y - 33}$$

where by a simple divisibility analysis (i.e. $2y-33$ must be a divisor of 330) we can obtain all the (fifteen) solutions of the given Diophantine equation.

Another example of this approach is the equation:

$$\frac{x}{8} + \frac{y}{5} - \frac{3}{z} = 7 \qquad\qquad (x, y, z \in \mathbb{Z},\ z \neq 0)$$

which can be reduced to the form:

$$z = \frac{120}{5x + 8y - 280}$$

and hence it is solved by divisibility analysis.

6.6.5 Conversion to Polynomial Equation

Some Diophantine equations involving fractions can be converted to polynomial equations by multiplying the entire equation by a suitable factor (and hence it is solved as a polynomial Diophantine equation; see § 6.1 and § 6.2). For example, the equation:

$$\frac{x}{y} + \frac{y}{x} = 2 \qquad\qquad (x, y \in \mathbb{Z},\ xy \neq 0) \tag{10}$$

can be solved by multiplying the equation by xy and rearranging to get $x^2 + y^2 - 2xy = 0$, i.e. $(x - y)^2 = 0$. Hence, we conclude that the general solution of the given equation is $(x, y) = (k, k)$ where $k \in \mathbb{Z}$ and $k \neq 0$.

In fact, this method applies to the more general version of Eq. 10, i.e.

$$\frac{x}{y} + \frac{y}{x} = z \qquad\qquad (x, y, z \in \mathbb{Z},\ xy \neq 0)$$

to prove that this equation has no solution except for $z = \pm 2$. This is achieved by forming the discriminant of the equation $x^2 - zxy + y^2 = 0$ (treated as a quadratic in x or as a quadratic in y; see § 6.2.6) where a detailed analysis of the discriminant will lead to the required conclusion (i.e. there is no solution except for $z = \pm 2$) as well as finding the solutions when $z = \pm 2$.

Another example is the equation:

$$\frac{x}{y} - \frac{y}{x} = 1 \qquad (x, y \in \mathbb{Z},\ xy \neq 0) \tag{11}$$

which can be solved by multiplying it by xy and rearranging to get $x^2 - xy - y^2 = 0$. On treating this as a one-variable quadratic equation in x (see § 6.2.6) and analyzing its discriminant we conclude that the given equation has no solution.

A similar method applies to the equation:

$$\frac{x}{y} - \frac{y}{x} = 2 \qquad (x, y \in \mathbb{Z},\ xy \neq 0)$$

which can be solved by multiplying it by xy and rearranging to get $x^2 = 2xy + y^2$ and hence:

$$2x^2 = x^2 + x^2 = x^2 + (2xy + y^2) = x^2 + 2xy + y^2 = (x + y)^2$$

On taking the square root of both sides we get: $x\sqrt{2} = \pm(x + y)$ which is impossible because $x\sqrt{2}$ is irrational while $(x + y)$ is an integer. So, we conclude that the given equation has no solution.

In fact, the method of Eq. 11 applies to the more general version of Eq. 11, i.e.

$$\frac{x}{y} - \frac{y}{x} = z \qquad (x, y, z \in \mathbb{Z},\ xy \neq 0)$$

to prove that this equation has no solution except for $z = 0$ where this is achieved by analyzing the discriminant of the equation $y^2 + zxy - x^2 = 0$ (which is obtained by multiplying the given equation by xy) as a quadratic in y (see § 6.2.6). The analysis will lead to the required conclusion as well as finding the solutions when $z = 0$.

Conversion to polynomial equation can also be used to solve the equation:

$$\frac{1}{x} + \frac{1}{y} = 1 \qquad (x, y \in \mathbb{Z},\ xy \neq 0)$$

where it is reduced to the form $(1 - x)(1 - y) = 1 = (-1)(-1)$ and hence its only solution (i.e. $x = y = 2$) can be obtained by factorization analysis (as indicated).

6.6.6 Comparison to a Similar Equation

Some equations can be easily solved by comparing them to similar equations whose solutions are known (or can be obtained easily). For example, the equation:

$$\frac{14}{x} - \frac{y}{19} = 25 \qquad (x, y \in \mathbb{Z},\ x \neq 0)$$

can be easily solved by comparing it to Eq. 8 where we write our equation as $\frac{14}{x} + \frac{Y}{19} = 25$ (with $Y = -y$) and hence we should have the same solutions for (x, Y) as those found for Eq. 8. The final solutions for our equation are then obtained by reversing the sign of Y.

This method of comparison similarly applies to the equation:

$$\frac{20}{x} - \frac{33}{y} = 2 \qquad\qquad (x, y \in \mathbb{Z},\ xy \neq 0) \tag{12}$$

which can be compared to Eq. 9 and hence its solutions are obtained from the solutions of Eq. 9 by reversing the sign of y. This is because if we write Eq. 12 as $\frac{20}{x} + \frac{33}{-y} = 2$ then it is no more than Eq. 9 with the sign of y being reversed.

6.6.7 Symmetry-Based Ordering and Magnitude Analysis

Some equations have a symmetry in their variables that can be exploited in an analysis based on ordering the magnitude of the variables. For example, the equations:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \qquad \text{and} \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 \qquad (x, y, z \in \mathbb{N})$$

can be investigated preliminarily under the ordering assumption that $x \leq y \leq z$ and hence their solutions (under this assumption) are obtained rather easily by exploiting the restrictions on the magnitude of the variables where these restrictions are based on the restriction on the magnitude of the LHS of these equations (which is imposed by their RHS) as well as the restriction imposed by the ordering assumption. The remaining solutions are then obtained by lifting the ordering assumption and hence permuting the variables in the *initial solutions* (i.e. the solutions obtained under the ordering assumption) where permuting the variables in the initial solutions is justified by the aforementioned symmetry. Hence, all the solutions are obtained in this two-stage process that exploits and employs symmetry, ordering and magnitude restrictions.

Also see § 10.

6.7 Equations Involving Roots and Fractions

The best approach for tackling this sort of equations is to linearize the radicals (i.e. the roots are replaced by non-radical symbols such as replacing $\sqrt{x}$ by X) and hence the problem is reduced in difficulty because we deal first with an equation that involves non-radical fractions while dealing with the issue of radicals is deferred to the end (where it is usually managed rather easily by raising the variables of the obtained solutions to suitable powers). For example, the equation:

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} = 2 \qquad\qquad (x, y, z \in \mathbb{N}) \tag{13}$$

can be solved by converting it first to the form:

$$\frac{1}{X} + \frac{1}{Y} + \frac{1}{Z} = 2 \qquad\qquad (X, Y, Z \in \mathbb{N}) \tag{14}$$

where the latter equation is solved as an equation involving fractions (see § 6.6). The final solutions (i.e. the solutions of Eq. 13) are then obtained by squaring the values of the variables of the solutions of Eq. 14. Many other similar equations can be dealt with by the same method.

6.8 Equations involving Factorials

A common approach in this type of problems is to obtain the solutions for the low values of factorial by inspection while the cases of high values of factorial are analyzed and tackled by a more general approach (e.g. by modular reduction). However, other (more versatile) tricks and techniques may be required or employed to solve this type of problems (e.g. when the factorial term is scaled by an integer factor). Some of these issues and details are demonstrated and clarified in the following examples:

1. The equation:

$$11x - y! = 13 \qquad\qquad (x \in \mathbb{Z},\ y \in \mathbb{N}^0)$$

has no solution for $y < 13$ (where this result is obtained by inspection). For $y \ge 13$ we have $11x \overset{13}{=} 0$ whose solution is $x = 13k$ where $k \in \mathbb{Z}$. On substituting this in the given equation and simplifying we get $k = \frac{(y!/13)+1}{11}$ which implies that k cannot be an integer and hence the given equation has no solution.

2. The equation:

$$x^2 - y! = 3 \qquad\qquad (x \in \mathbb{Z},\ y \in \mathbb{N}^0)$$

has no solution for $y > 3$ because $x^2 \overset{4}{=} 3$ (which we obtain by reducing the given Diophantine equation in modulo 4) has no solution (since 3 is not a quadratic residue of 4). Hence, if there is any solution then we must have $y = 0, 1, 2, 3$. So, by inspecting these values of y we get all the solutions of the given equation, i.e. $(x, y) = (\pm 2, 0)$, $(\pm 2, 1)$, $(\pm 3, 3)$.

The equation:

$$x^2 - y! = 2 \qquad\qquad (x \in \mathbb{Z},\ y \in \mathbb{N}^0)$$

can be similarly analyzed and solved (where reduction in modulo 3 is used for $y > 2$ with inspection of $y = 0, 1, 2$).

3. The equation:

$$16x + 9y! = 121 \qquad\qquad (x \in \mathbb{Z},\ y \in \mathbb{N}^0)$$

has no solution for $y > 1$ (due to parity violation) and hence it is solvable only for $y = 0, 1$ (i.e. $x = 7$ corresponding to $y = 0, 1$).

4. The equation:

$$x^2 + y^2 - z! = 3 \qquad\qquad (x, y \in \mathbb{Z},\ z \in \mathbb{N}^0)$$

can be solved by noting that for $z > 3$ we have $x^2 + y^2 \stackrel{4}{=} 3$ which has no solution and hence all we need to do is to inspect $z = 0, 1, 2, 3$ which lead to its (twenty) solutions.

5. The equation:

$$x^2 + y^2 + z! = 24 \qquad\qquad (x, y \in \mathbb{Z},\ z \in \mathbb{N}^0)$$

can be solved by noting that z cannot be greater than 4 and hence all we need to do is to inspect $z = 0, 1, 2, 3, 4$ which lead to its (five) solutions.

6. The equation:

$$3x + 5y + 15z! = 17 \qquad\qquad (x, y \in \mathbb{Z},\ z \in \mathbb{N}^0)$$

can be solved by reducing it in modulo 5 to get $x = 4 + 5k$ and by reducing it in modulo 3 to get $y = 1 + 3s$. On substituting these into the original equation we get its (parameterized) solutions.

7. The equation:

$$2x + 3y - 6z! = 222 \qquad\qquad (x, y \in \mathbb{Z},\ z \in \mathbb{N}^0)$$

can be solved by reducing it in modulo 3 to get $x = 3k$ and by reducing it in modulo 2 to get $y = 2s$. On substituting these into the original equation we get its (parameterized) solutions.

6.9 Trigonometric Equations

The simple (and most common types) of trigonometric Diophantine equations can be solved rather easily by investigating the cyclic behavior of the terms of the equation and considering all the combinations (of the values of these terms) that satisfy the given equation. The following examples should give a reasonable insight about how to tackle and solve (the common types) of trigonometric Diophantine equations:

1. The equation:

$$3\sin\left(\frac{x\pi}{2}\right) - 4\sin\left(\frac{y\pi}{2}\right) = 1 \qquad\qquad (x, y \in \mathbb{Z})$$

can be solved by noting that the sine function of integer multiples of $\pi/2$ takes the values $0, 1, 0, -1$ corresponding to $x \stackrel{4}{=} 0, 1, 2, 3$ (and similarly for y), and hence the only possibility for the LHS of this equation to be equal to 1 is when $\sin\left(\frac{x\pi}{2}\right) = \sin\left(\frac{y\pi}{2}\right) = -1$ so that the LHS becomes $3(-1) - 4(-1) = 1$. Accordingly, the solutions of the given equation are $(x, y) = (3 + 4k,\ 3 + 4s)$ where $k, s \in \mathbb{Z}$.

2. The equation:

$$5\sin\left(\frac{x\pi}{2}\right) + \cos(y\pi) = 3 \qquad\qquad (x, y \in \mathbb{Z})$$

can be solved by noting that the sine function of integer multiples of $\pi/2$ takes the values $0, 1, -1$ while the cosine function of integer multiples of π takes the values $1, -1$. Therefore, the term $5\sin\left(\frac{x\pi}{2}\right)$ takes the values $0, 5, -5$ while the term $\cos(y\pi)$ takes the values $1, -1$. As we see, there is no combination of these values that can make the sum of the terms on the LHS to be equal to 3. So, the given equation has no solution.

3. The equation:

$$2\sin\left(\frac{x\pi}{2}\right) + 3\cos\left(\frac{y\pi}{2}\right) = 2 \qquad\qquad (x, y \in \mathbb{Z})$$

can be solved (like the previous equation) by noting that $2\sin\left(\frac{x\pi}{2}\right) = 0, 2, 0, -2$ corresponding to $x \stackrel{4}{=} 0, 1, 2, 3$ while $3\cos\left(\frac{y\pi}{2}\right) = 3, 0, -3, 0$ corresponding to $y \stackrel{4}{=} 0, 1, 2, 3$. So, their sum is equal to 2 when $(x, y) \stackrel{4}{=} (1, 1)$ and $(x, y) \stackrel{4}{=} (1, 3)$. Hence, the solutions of the given equation are $(x, y) = (1 + 4k,\ 1 + 2s)$ where $k, s \in \mathbb{Z}$.

4. The equation:

$$\sin\left(\frac{x\pi}{2}\right) + 5\cos(y\pi) = 6\cos(z\pi) \qquad\qquad (x, y, z \in \mathbb{Z})$$

can be solved by noting that:

$$\sin\left(\tfrac{x\pi}{2}\right) = 0, 1, 0, -1 \quad \text{for} \quad x \stackrel{4}{=} 0, 1, 2, 3$$

$$5\cos(y\pi) = 5, -5 \quad \text{for} \quad y \stackrel{2}{=} 0, 1$$

$$6\cos(z\pi) = 6, -6 \quad \text{for} \quad z \stackrel{2}{=} 0, 1$$

So, the two sides become equal in the following two cases (which represent the solutions of this equation):

$$(x, y, z) = (1 + 4k,\ 2s,\ 2t) \qquad\qquad (x, y, z) = (3 + 4k,\ 1 + 2s,\ 1 + 2t)$$

where $k, s, t \in \mathbb{Z}$.

5. The equation:

$$\tan\left(x\pi + \frac{\pi}{3}\right) + 2\sin\left(y\pi + \frac{2\pi}{3}\right) = 2\cos\left(\frac{z\pi}{6}\right) \qquad (x, y, z \in \mathbb{Z})$$

can be solved by noting that:

$\tan\left(x\pi + \frac{\pi}{3}\right) = \sqrt{3}$ for all $x \in \mathbb{Z}$

$2\sin\left(y\pi + \frac{2\pi}{3}\right) = \sqrt{3}, -\sqrt{3}$ for $y \overset{2}{=} 0, 1$

$2\cos\left(\frac{z\pi}{6}\right) = 2, \sqrt{3}, 1, 0, -1, -\sqrt{3}, -2, -\sqrt{3}, -1, 0, 1, \sqrt{3}$ for $z \overset{12}{=} 0, 1, 2, \ldots, 11$

So, the two sides become equal in the following two cases:

$$(x, y, z) = (k,\ 1 + 2s,\ 3 + 12t) \qquad (x, y, z) = (k,\ 1 + 2s,\ 9 + 12t)$$

which can be combined in the following formula (which represents all the solutions of the given equation): $(x, y, z) = (k,\ 1 + 2s,\ 3 + 6t)$ where $k, s, t \in \mathbb{Z}$.

7 Reduction and Analysis by Modular Arithmetic

Reduction and analysis by modular arithmetic is very common method for solving various types of Diophantine equations, and hence considering modular arithmetic as an aiding technique for solving Diophantine equations is highly recommended especially when dealing with polynomial and exponential Diophantine equations. In fact, we already discussed this method within the context of its use in solving various types of Diophantine equations (see for instance § 6.2.4 and § 6.3.2). So, in this section we discuss some general issues about this method (namely its logical and mathematical foundations, its purposes and some recommendations and guidelines about its use and employment highlighting finally its importance for non-solvability).

7.1 Logical and Mathematical Foundations

Regarding the logical and mathematical foundations of the use of modular arithmetic in solving Diophantine equations, we refer the reader to § § 2.7.6 of [2] (also see § 8.1 of [3]) where we discussed this subject[18] in detail. So, all we need to know here are the following simple rules:[19]

• If $f(x,y) = 0$ then $f(x,y) \stackrel{m}{=} 0$ for *any* $\mathbb{N} \ni m > 1$, and hence (by contraposition) if $f(x,y) \underset{}{\stackrel{m}{\neq}} 0$ for a *specific* m then $f(x,y) \neq 0$.

• If $f(x,y) \stackrel{m}{=} 0$ for all $\mathbb{N} \ni m > 1$ then $f(x,y) = 0$.

7.2 Purposes of Modular Reduction and Analysis

The main (or general) purpose of using modular reduction and analysis as an aiding tool for solving Diophantine equations is to reduce the effort (required for searching for solution) by exploiting the versatile collection of rules (or techniques or ...) of modular arithmetic and its rather simplified mathematical machinery (such as the possible use of small numbers or the classification of integers in a rather simple and well organized residue systems). However, there are many specific purposes that come under this general purpose. In the following list, we try to outline some of the common specific purposes (or benefits or objectives etc.) of using modular reduction and analysis in solving Diophantine

[18] We mean the subject of relationship between the ordinary (Diophantine) equation and the corresponding congruence equation, e.g. the relationship between the Diophantine equation $x^2 + 3xy = 0$ and the congruence equation $x^2 + 3xy \stackrel{3}{=} 0$.

[19] In the following, $f(x,y)$ represents a Diophantine expression (like $x^2 + 3xy$) in two variables noting that this applies to Diophantine expressions in more than two variables (e.g. $x^2 + 3xy - 5z^3$).

equations where we mostly rely for explanation and clarification on the examples that we discussed and investigated elsewhere:

1. **Showing non-solvability** of the Diophantine equation (see for instance the examples of Eqs. 1 and 3).
2. **Removing some variables** and hence simplifying the analysis of the given equation (see for instance the example of Eq. 2).
3. **Revealing patterns and clues** that can help in solving the Diophantine equation (see for instance the examples of Eqs. 4 and 5).

We note in this regard that the employed modular reduction may not introduce any "visible" change on the original Diophantine equation apart from converting it from an ordinary equation to a congruence equation.

7.3 Recommendations and Guidelines

We already discussed the main recommendations and guidelines about the use of modular reduction and analysis during our discussion of specific types of Diophantine equations (see for instance § 6.2.4 and § 6.3.4). So, in the following points we just outline and summarize what we have given earlier:

- Try to find and use small reduction moduli when possible.
- Try to find and use a prime numbers (as a moduli) when possible.
- Consider using more than one reduction modulo in the analysis.
- Remember that modular reduction and analysis is primarily a tool for showing non-solvability (see the rules of § 7.1 as well as § 7.4). However, it is commonly used (with further analysis and extra effort) for investigating and finding the solutions of the Diophantine equations (if they exist) by imposing certain restrictions and conditions on the solutions and showing their types and forms (such as being odd or divisible by 5).

7.4 Importance of Modular Analysis in Proving Non-Solvability

It is important to note that modular reduction and analysis is especially important in showing and proving that a given Diophantine equation has no solution (either conditionally or unconditionally). In fact, it is usually the only (or at least the main) possible tool/method for doing this job and achieving this purpose. Therefore, as soon as we have a guess or a hint or an indication (e.g. from initial computational investigation; see § 5) that a given equation has no solution, we should recall our modular arithmetic techniques and skills to find a modulo in which the given Diophantine equation has no solution when it is reduced in that modulo. In fact, parity analysis and check (which is one of the most common and basic methods for establishing non-solvability; see for instance 2.1) is no more than modular reduction and analysis in modulo 2 (where its wide spread use and its distinction from modular analysis is because of its simplicity and intuitivity as well as other reasons and factors).

8 Factorization Analysis

Factorization analysis is a very common and useful method for analyzing and solving Diophantine equations and hence it should always be considered as one of the first approaches in tackling Diophantine problems. In its most common and simple form (noting that it has various forms and variants as will be outlined next) factorization analysis is based on our ability to produce a factored expression involving variables (on the LHS) that is equal to a specific number (on the RHS). The number of the possible factors on the two sides may also be considered in some variants of factorization analysis since it can eliminate certain possibilities for the solution. Other factors and considerations (such as divisibility or primality/composity) may also be considered and included in more versatile factorization analysis arguments.

In the following subsections we will briefly investigate the aforementioned variants of factorization analysis.

8.1 Simple Factorization Analysis

As indicated in the preamble of this section, simple factorization analysis is based on producing a factored expression involving variables (on the LHS) that is equal to a specific number (on the RHS) where the factors involving variables (on the LHS) can be matched with numeric factors (on the RHS) to produce systems of simultaneous equations that can be solved to produce the solution(s) of the given problem. In fact, some examples of this type of factorization analysis have already been given (see for instance the examples of point 1 of § 6.2.5).

8.2 Factorization Analysis based on the Number of Factors

Factorization analysis may also be based on comparing the number of factors (i.e. not their specific form or value) on the two sides of the equation. For example, the function:

$$f(x, y) = 120x^5 + 274x^4y + 225x^3y^2 + 85x^2y^3 + 15xy^4 + y^5$$

can be factorized in the following 5-factor form:

$$f(x, y) = (x + y)(2x + y)(3x + y)(4x + y)(5x + y)$$

and hence we can conclude (through factorization analysis based on the number of factors on the two sides) that $f(x, y) = 21$ has no solution (because 21 can be factorized only in 2 or 3 or 4 distinct integer factors) while $f(x, y) = 45$ can have (and actually has) solution (because 45 can be factorized into 5 distinct integer factors). However, it should be noticed that this type of factorization analysis is primarily for proving non-solvability and hence solvability (and obtaining the solutions) requires extra work (noting that having the same number of factors on the two sides of equation is a necessary but not sufficient condition for solvability in this type of factorization analysis).

8.3 Factorization with Divisibility Analysis

In this type of factorization analysis we consider employing divisibility arguments of one side (or of factors of one side) by the other side (or by factors of the other side). In fact, some examples of this type of factorization analysis have already been given (see for instance the examples of points 2 and 3 of § 6.2.5).

8.4 Factorization with Primality/Composity Analysis

In this type of factorization analysis we consider employing primality/composity arguments where one side (or factors of one side) is compared to the other side (or to factors of the other side) from this perspective (i.e. being prime or composite). Some simple examples of this type of factorization analysis have been given in § 2.2.

9 Comparison to Similar Equations and Problems

An ideal way for solving a given Diophantine equation is to compare it to a similar equation whose solutions are known (or whose solutions are easier to obtain) and hence obtain the solutions of the given equation with no effort (or with minimal effort). In fact, sometimes solving a given Diophantine equation may require no more than copying and pasting the solution of a previously-solved problem with some modifications and adaptations to reflect the specific characteristics of the given Diophantine equation. Therefore, it is recommended when tackling a Diophantine equation to consider this method of solution as one of the first options to recall and consider.

As hinted above, the advantage of the method of comparison is simplicity and ease where the effort required to solve the given Diophantine equation is reduced substantially. However, an obvious limitation of this method is that we need to have a similar equation with known (or easy-to-obtain) solutions which is obviously not available in most cases. However, it is useful to search for such an equation in our collection of previously-solved Diophantine equations, and hence keeping an organized and classified "database" of Diophantine equations (with known solutions) is very useful (especially to those who specialize in the subject of Diophantine equations).[20] Some examples of this method are given in the following:

1. Let us assume that we already investigated the Diophantine equation:

$$5x^3 + 4y^3 - 9 = 0 \tag{15}$$

and obtained its solutions which are $(x, y) = (1, 1)$ and $(x, y) = (13, -14)$. Now, if we have to solve the Diophantine equation:

$$5x^3 - 4y^3 + 9 = 0 \tag{16}$$

then all we need to do is to change the sign of the x value of the solutions of Eq. 15 and hence obtain the solutions $(x, y) = (-1, 1)$ and $(x, y) = (-13, -14)$. This is because if we multiply Eq. 16 by -1 then we obtain:

$$5(-x)^3 + 4y^3 - 9 = 0$$

which is no more than Eq. 15 with the sign of x being reversed.

Similarly, if we have to solve the Diophantine equation:

$$5x^3 + 4y^3 + 9 = 0 \tag{17}$$

[20] In fact, this is one reason for organizing and classifying Diophantine equations in our books (see [2, 3]).

then all we need to do is to change the sign of the x and y values of the solutions of Eq. 15 and hence obtain the solutions $(x, y) = (-1, -1)$ and $(x, y) = (-13, 14)$. This is because if we multiply Eq. 17 by -1 then we obtain:

$$5(-x)^3 + 4(-y)^3 - 9 = 0$$

which is no more than Eq. 15 with the sign of x and y being reversed.

2. Let us assume that we already investigated the Diophantine equation:

$$x^3 + y^3 + x^2 + y^2 = 0 \tag{18}$$

and obtained its solutions which are $(x, y) = (0, 0)$, $(0, -1)$, $(-1, 0)$, $(-1, -1)$. Now, if we have to solve the Diophantine equation:

$$x^3 + y^3 - x^2 - y^2 = 0 \tag{19}$$

then all we need to do is to change the sign of the x and y values of the solutions of Eq. 18 and hence obtain the solutions $(x, y) = (0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. This is because if we multiply Eq. 19 by -1 then we obtain:

$$(-x)^3 + (-y)^3 + x^2 + y^2 = 0 \qquad \text{i.e.} \qquad (-x)^3 + (-y)^3 + (-x)^2 + (-y)^2 = 0$$

which is no more than Eq. 18 with the sign of x and y being reversed.

3. We should also refer to the examples of § 6.4.3 and § 6.6.6 where we used the method of comparison for solving some Diophantine equations of certain types.

We should finally note that comparison to similar equations and problems may require some manipulations and transformations (see the examples in § 4). In fact, we may even extend the method of comparison beyond direct comparison of two similar equations with a specific form and hence we may consider comparing equations of certain characteristic features (though they may not look similar in form) that make their method of solution (or the rationale and logic behind their method of solution) similar.

10 Symmetries and Cycling Patterns in the Variables

Symmetry in the variables[21] of a Diophantine equation is a useful feature that can be exploited in assuming temporarily that the variables have a certain increasing or decreasing order. This should facilitate the search for a solution where the final and complete solution can be obtained eventually by permuting the solution obtained on the base of ordering assumption. So, when tackling a Diophantine equation with total or partial symmetry in its variables it is recommended to consider such a symmetry by synthesizing a logical or mathematical argument that exploits such a symmetry.

We should also look for any cyclic pattern[22] in the variables of Diophantine equations where this pattern can be exploited similarly, i.e. by assuming initially that a certain variable is the biggest or the smallest (since cycling would allow us to bring this variable to a certain position in the equation where ordering can be exploited to make an argument that leads to the interim solution and this solution can be generalized later by lifting the condition of ordering).

The following are some examples of how symmetry and cyclic pattern in the variables can be exploited in solving Diophantine equations:

1. The Diophantine equation:

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{3} \qquad\qquad (x, y \in \mathbb{N})$$

is symmetric in x, y and hence we can assume initially that $x \le y$. Accordingly, we can argue (based on this assumption and considering the magnitude of the RHS as well as similar factors) that x must be either 2 (and hence $y = 6$) or 3 (and hence $y = 3$). So, we obtain the interim solutions $(x, y) = (2, 6)$ and $(3, 3)$. Now, if we lift the condition $x \le y$ (thanks to the symmetry) we obtain (by permutation) another solution, i.e. $(x, y) = (6, 2)$. So, the given Diophantine equation has only three solutions.

2. The Diophantine equation:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 \qquad\qquad (x, y, z \in \mathbb{N})$$

is symmetric in x, y, z and hence we can assume initially that $x \le y \le z$. Accordingly, we can argue (based on this assumption) that x cannot be greater than 1 and y must be 2

[21] "Symmetry" in the variables of an algebraic expression means that the variables can be exchanged without affecting the expression.

[22] "Cyclic pattern" in the variables of an algebraic expression means that the variables can be exchanged in a certain cyclic order without affecting the expression although the expression is affected if the variables are exchanged without regard to the cyclic order. So, cyclic pattern is a restricted form of symmetry.

and hence z must be 2. This argument produces the interim solution $(x, y, z) = (1, 2, 2)$ which can be generalized by permutation (thanks to the symmetry) to produce the other two solutions, i.e. $(x, y, z) = (2, 1, 2)$ and $(2, 2, 1)$.

3. The Diophantine equation:

$$x + y + z = xyz \qquad\qquad (x, y, z \in \mathbb{Z},\ xyz \neq 0)$$

is symmetric in x, y, z and hence we can assume initially that $|x| \leq |y| \leq |z|$. Accordingly, we can argue (based on this assumption) that x must be equal to ± 1 and hence either $y = -2$ and $z = -3$ (i.e. when $x = -1$) or $y = 2$ and $z = 3$ (i.e. when $x = 1$). This argument produces the two interim solutions $(x, y, z) = \pm(1, 2, 3)$ which can be generalized by permutation (thanks to the symmetry) to produce the other ten solutions, i.e. $(x, y, z) = \pm(1, 3, 2)$, $\pm(2, 1, 3)$, $\pm(2, 3, 1)$, $\pm(3, 1, 2)$, $\pm(3, 2, 1)$.

4. The Diophantine equation:

$$x^2 + y^2 = z^2 \qquad\qquad (x, y, z \text{ are consecutive natural numbers})$$

is symmetric in x, y and hence we can assume initially that $x < y$ (i.e. $y = x+1$), that is:
$x^2 + (x + 1)^2 = (x + 2)^2 \quad \rightarrow \quad x^2 - 2x - 3 = 0 \quad \rightarrow \quad (x + 1)(x - 3) = 0$
i.e. $x = 3$ (noting that $x \in \mathbb{N}$). Therefore, we have only one interim solution, i.e. $(x, y, z) = (3, 4, 5)$. Now, if we lift the condition $x < y$ (thanks to the symmetry) we obtain (by permutation) the other solution, i.e. $(x, y, z) = (4, 3, 5)$.

5. The Diophantine equation:

$$x + y + z + w = xyzw \qquad\qquad (x, y, z, w \in \mathbb{N})$$

is symmetric in x, y, z, w and hence we can assume initially that $x \leq y \leq z \leq w$. Accordingly, we can argue as before (based on this assumption) that we must have $x = y = 1$ and $z = 2$ and hence $w = 4$. So, the interim solution is $(x, y, z, w) = (1, 1, 2, 4)$ which can be generalized by permutation (thanks to the symmetry) to produce the other eleven solutions.

6. The Diophantine equation $x^4 + y^4 + z^4 = 3042$ (which will be discussed in § 11) is another example of symmetry in the variables.

7. The Diophantine equation:

$$x^3y + y^3z + z^3x = 1 \qquad\qquad (x, y, z \in \mathbb{N}^0)$$

is not symmetric in its variables, e.g. if we exchange x and y we get:

$$y^3x + x^3z + z^3y = 1$$

which is not the same as the original equation. However, it is cyclic in its variables, i.e. if we exchange $x \to y$, $y \to z$ and $z \to x$ we get:

$$y^3 z + z^3 x + x^3 y = 1$$

which is the same as the original equation (apart from the order of terms which is irrelevant in this context). So, we can assume initially that $x \leq y$ and $x \leq z$ (since we can cycle the variables to put x in such a position in the equation noting that we cannot assume that $x \leq y \leq z$ because such an ordering flexibility requires full symmetry). Accordingly, we can argue (based on this assumption) that x must be 0 (because otherwise the LHS will be greater than 1) and hence we must have $y = z = 1$. So, the interim solution is $(x, y, z) = (0, 1, 1)$. Now, if we cycle the variables we obtain the other two solutions, i.e. $(x, y, z) = (1, 0, 1)$ and $(x, y, z) = (1, 1, 0)$.

11 Upper and Lower Bounds

Imposing upper or/and lower bounds or limits (usually in the form of bounding inequalities) on the potential solutions should be considered when tackling Diophantine equations (e.g. when dealing with equations involving fractions). This can reduce the complexity of the process of search for solution substantially since it reduces the number of possible solutions. In fact, it can reduce this number from being infinite to finite and this should allow the use of more simple methods of inspection and search (e.g. by computational tools and techniques).

The two main considerations for imposing such bounds are:

• Magnitude[23] considerations where the potential solutions cannot exceed in magnitude certain (upper or/and lower) limits. Bounds imposed by magnitude considerations can impose a limit on the number of possible solutions, i.e. they can make this number finite when both lower and upper bounds are imposed.

• Sign considerations where the potential solutions can be of only one type of sign (i.e. positive or negative). Although bounds imposed by sign considerations should reduce the difficulty of the problem they usually do not impose a limit on the number of possible solutions, i.e. this number remains infinite. In fact, sign considerations can be seen as a special form of magnitude considerations (which we discussed in the previous point) where only upper/lower bound is imposed.

The following are a few examples for the use of bounds (and bounding arguments) in solving Diophantine problems:

1. Consider the following Diophantine equation:

$$\frac{1}{x} + \frac{1}{y} = z \qquad\qquad (x, y, z \in \mathbb{Z},\ xy \neq 0)$$

It should be obvious that the LHS (and hence z) cannot be less than -2 or greater than $+2$ and hence:

• If $z = -2$ then $(x, y, z) = (-1, -1, -2)$.

• If $z = -1$ then $(x, y, z) = (-2, -2, -1)$.

• If $z = 0$ then $(x, y, z) = (k, -k, 0)$ where $\mathbb{Z} \ni k \neq 0$.

• If $z = 1$ then $(x, y, z) = (2, 2, 1)$.

• If $z = 2$ then $(x, y, z) = (1, 1, 2)$.

[23] "Magnitude" here should mean the position on the number line rather than the absolute value (as we usually use).

2. Consider the following Diophantine equation:

$$x^4 + y^4 + z^4 = 3042 \qquad (x, y, z \in \mathbb{Z})$$

It should be obvious that any one of the three variables cannot be less than -7 or greater than $+7$ (because otherwise the LHS will be greater than 3042), and hence if we carry a simple computational search within the limits $-7 \le x, y, z \le +7$ we will find all the (forty eight) solutions of this equation.

3. The following Diophantine equation:

$$x^2 - xy + y^2 + 2x - y = 2 \qquad (x, y \in \mathbb{Z})$$

can be put in the following form:

$$(x + 2)^2 + (y - 1)^2 + (x - y)^2 = 9$$

and hence we must have $-3 \le (x + 2) \le 3$, i.e. $-5 \le x \le 1$. Now, if we investigate all the seven possibilities (i.e. $x = -5, -4, -3, -2, -1, 0, 1$) we get all the (six) solutions of the given equation.

4. Consider the following Diophantine equations:

$$2^x + 3^y = 1 \qquad 4^x + 9^y = 2 \qquad (x, y \in \mathbb{N}^0)$$

It should be obvious that the first equation has no solution because its LHS cannot be less than 2, while the second equation has only the trivial solution $x = y = 0$ because otherwise the LHS will be greater than 2.

5. Consider the following Diophantine equations:

$$\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 7 \qquad \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 2 \qquad (x, y, z \in \mathbb{N})$$

It should be obvious that these equations have no solution because the LHS of the first equation cannot exceed 6 while the LHS of the second equation must exceed 2.

6. Consider the following Diophantine equation:

$$z = \frac{1}{x} + \frac{1}{y} + \frac{2}{xy} \qquad (x, y, z \in \mathbb{N})$$

It should be obvious that the RHS (and hence z) cannot exceed 4. By a similar magnitude argument it can be easily shown that x and y cannot exceed 4. So, all we need to do is to inspect (computationally) the combinations of $x, y, z = 1, 2, 3, 4$ to obtain the (five) solutions of the given equation.

7. Consider the following system of Diophantine equations:

$$3xy + 5x^3y = 230 \qquad \text{and} \qquad x^2 + xy = 14 \qquad (x, y \in \mathbb{Z}) \tag{20}$$

From the first equation it is fairly obvious that x and y must have the same sign (because otherwise the LHS will be non-positive). Now, from the second equation it should be obvious that the absolute value of x cannot exceed 3 and hence we have only the following 6 possibilities to consider: $x = -3, -2, -1, 1, 2, 3$. On inspecting these possibilities we find the solutions of this system, i.e. $(x, y) = (-2, -5)$ and $(x, y) = (2, 5)$.

12 Reduction of Domain

When dealing with Diophantine equations it is recommended to consider reducing the domain of solution temporarily until a solution is found (for the reduced domain) where this solution can be extended and generalized later on to reach the final and complete solution for the entire domain. For example, if we are dealing with a Diophantine equation in the domain of integers $\mathbb{Z}$ involving variables with even powers then we can start by considering its solution in the domain of natural numbers $\mathbb{N}$ (instead of $\mathbb{Z}$) where the final and complete solution can be obtained later on by extending the domain to the negative integers (noting that even powers do not distinguish between positive and negative bases).

The reduction of domain may also take the form of splitting the domain into parts and dealing with these parts separately (and usually independently). For example, we may consider splitting[24] the integer domain of a Diophantine equation to negative integers and positive integers where we investigate these sub-domains separately (e.g. by applying different arguments and techniques to each sub-domain) and the final solution (over the entire integer domain) will be obtained in the end by taking the union of the solutions in the sub-domains (noting the possibility of having no solution in some or all sub-domains). Such a scenario may also be considered by splitting the domain to odd integers and even integers (or to integers greater than and integers less than a certain value or ...) where we deal with these sub-domains separately and obtain the entire solution in stages (as before).

Anyway, reduction/splitting of domain in its various shapes and forms is a very common approach in tackling and solving Diophantine problems and hence it should always be considered since it usually brings many benefits (such as reducing the complexity of the given problem and making the strategy of tackling it tidy and organized) and can provide key clues for solving it. It can also provide a partial solution to the given problem, i.e. when the investigation leads to solution only in some parts of the domain (where this partial solution can be extended in the future to include the entire domain or where the original problem is modified to consider such a restriction on the domain).

We present in the following some examples for the reduction of domain in tackling Diophantine problems:

1. The "Pythagorean equation" $x^{2a} + y^{2b} = z^{2c}$ (where $x, y, z \in \mathbb{Z}$ and $a, b, c \in \mathbb{N}$) is an obvious example for the possibility of considering the reduction of domain where we can

[24] Such a split can involve some or all variables (depending on the nature of the Diophantine problem and the tackling strategy).

start by considering initially the solutions in the reduced domain of natural numbers (or non-negative numbers). Any solution obtained in this restricted domain can then be extended to the domain of integers by considering all the possible sign alterations in the values of the variables in the natural solutions or non-negative solutions).

2. Another obvious example for the reduction of domain is the Diophantine equations of the Pell type (see § 6.2.3) where we can obtain the solutions in the reduced domain of natural numbers first (e.g. by the use of standard Pell's techniques) and extend these solutions subsequently to the domain of integers by considering all the possible sign alterations in the values of the variables in the natural solutions.
3. Consider the following Diophantine equation:

$$x^3y + 3xy^3 + 7xy = 1085 \qquad\qquad (x, y \in \mathbb{Z})$$

It is fairly obvious that x and y have the same sign (because otherwise the LHS will be non-positive). Now, if we start by assuming that $x, y \in \mathbb{N}$ then we can easily obtain the solution $(x, y) = (1, 7)$. Following this, we extend this reduced domain (by lifting the condition $x, y \in \mathbb{N}$) to obtain the other solution, i.e. $(x, y) = (-1, -7)$ noting that the equation does not change by reversing the sign of x and y.
4. Consider the following Diophantine equation:

$$\frac{x}{y} + xy = 2 \qquad\qquad (x, y \in \mathbb{Z},\ y \neq 0)$$

It is obvious that x and y have the same sign (because otherwise the LHS will be non-positive). Now, if we start by assuming that $x, y \in \mathbb{N}$ then we can easily obtain the solution $(x, y) = (1, 1)$. Following this, we extend this reduced domain (by lifting the condition $x, y \in \mathbb{N}$) to obtain the other solution, i.e. $(x, y) = (-1, -1)$ noting that the equation does not change by reversing the sign of x and y.
5. Consider the system of Eq. 20. Noting that x and y must have the same sign, we can start by assuming that $x, y \in \mathbb{N}$ where we can easily obtain the solution $(x, y) = (2, 5)$. We then extend this reduced domain (by lifting the condition $x, y \in \mathbb{N}$) to obtain the other solution, i.e. $(x, y) = (-2, -5)$.[25]

[25] Noting that x and y have the same sign (as well as x^2 in the second equation), it should be obvious that the equations of this system do not change by such a change in sign and hence the second (i.e. negative) solution can be obtained by just reversing the sign of the first (i.e. positive) solution.

13 Basic Rules and Principles

When we deal with a Diophantine problem we should always try to recall common rules and basic principles that can help in solving the problem and reducing the amount of work required to solve it (since such rules and principles usually summarize and encompass a number of steps in the formal proof or argument required for solving the problem). In fact, we should even consider manipulating the given Diophantine equations in such a way that enables us to exploit such rules and principles. Prominent examples of such rules and principles are the (integer) ordering rules, the algebraic rules of (integer) powers, and the rules of primality and coprimality. These rules and principles are particularly useful for identifying the Diophantine equations and systems that have no solution (or have restrictions on their solutions).

A few examples of these rules and principles are given in the following points (noting that there are many other rules and principles like these that can be exploited for tackling Diophantine problems):

• No (perfect) square can be between consecutive (perfect) squares, no (perfect) cube can be between consecutive (perfect) cubes, and so on.
• No (perfect) square can be the sum of two odd squares.[26]
• Any (perfect) cube can be the difference of two (perfect) squares.
• The difference between two consecutive cubes cannot be divisible by 5.
• Any integer can be the sum of two perfect squares minus a perfect square.
• Any natural number can be the sum of four non-negative integer squares.
• The difference between two non-trivial perfect squares cannot be 4.
• If the square root of an integer is rational then the integer is a perfect square (i.e. the square root is an integer).
• Any two consecutive integers are coprime.
• The natural powers of distinct primes are coprime.
• Any odd prime is congruent (in modulo 4) either to $+1$ or to -1 (or similarly to $+3$).
• In any three consecutive odd integers exactly one of them is divisible by 3.
• All factorials are even except $0!$ and $1!$.
• The factorial $n!$ is not divisible by any prime $p > n$.
• All binomial coefficients and multinomial coefficients are integers (despite their appearance as ratios or fractions).

[26] This rule is actually based on the rules and properties of Pythagorean triples (see § 6.2.2) which will be mentioned later in this list.

• The rules (and properties) of Pythagorean triples should also be considered in this regard.
• Fermat's last theorem (see § 6.2.1) can also be considered in this regard (since it can be seen as a rule that eliminates the possibility of existence of solution or the possibility of existence of certain types of solution).

We present in the following some examples for the use of such rules and principles in tackling and solving Diophantine problems:

1. By simple algebraic manipulation, the following Diophantine equation:
$$4x^2 + 16y^2 - 9z^2 - 12x + 8y + 10 = 0 \qquad\qquad (x, y, z \in \mathbb{Z})$$
can be put in the following form:
$$(2x - 3)^2 + (4y + 1)^2 = (3z)^2$$
Now, if we recall the rule that "the sum of two odd squares cannot be a perfect square" then we can conclude that this Diophantine equation has no solution (i.e. the problem is solved with virtually no effort thanks to this rule).
2. The following Diophantine equation:
$$216x^3 + 27y^3 + 216x^2 + 72x - 721 = 0 \qquad\qquad (x, y \in \mathbb{N})$$
can be put in the following form:
$$(6x + 2)^3 + (3y)^3 = 9^3$$
and hence it can be solved by Fermat's last theorem, i.e. it has no solution in $\mathbb{N}$ (and in fact it has no solution even in $\mathbb{Z}$).
3. The following Diophantine equation:
$$16x^4 + 81y^4 = z^4 \qquad\qquad (x, y, z \in \mathbb{Z})$$
can be written as:
$$(2x)^4 + (3y)^4 = z^4$$
and hence by Fermat's last theorem we can conclude that it can have only trivial solutions (i.e. solutions with $xyz = 0$). This restriction on the solutions should reduce the difficulty of searching for solutions substantially.
4. Consider the following system of Diophantine equations:
$$y^3 + 3y^2 - x^2 + 3y - z^5 + z + 1 = 0 \qquad \text{and} \qquad x^2 - y^3 = 0 \qquad\qquad (x, y, z \in \mathbb{Z})$$

If we put the equations of this system in the following forms:

$$(y+1)^3 - x^2 = z^5 - z \qquad \text{and} \qquad x^2 = y^3$$

and substitute from the second equation into the first equation then we get:

$$(y+1)^3 - y^3 = z^5 - z$$

Now, if we remember the rule that "the difference between two consecutive cubes cannot be divisible by 5" and note that $z^5 - z$ is divisible by 5 then we can conclude that this system has no solution.

14 Other Recommendations

There are many other recommendations that should be considered when dealing with Diophantine problems (noting that many of these recommendations usually depend on the particular type of Diophantine equation/system of concern). Examples of these recommendations are:

1. Recall interesting theorems (such as Wilson's theorem and Fermat's little theorem) possibly as part of modular arithmetic investigation and analysis to the Diophantine equation. For example, Wilson's theorem may be useful to recall and use (within modularity investigation and analysis) when the Diophantine problem involves factorials.
2. Consider special (or limiting or obvious or eccentric or ...) cases and instances such as when one (or more) of the variables is 0 or ± 1 or goes to infinity or becomes negative. Such considerations can give an insight in the solution (or reduce the possibilities or organize the approach of solution or give a clue to the solution or ...).
3. Give special attention to the dominating terms in the equation(s) which can (for instance) impose limits or determine the eventual tendency of the equation(s).
4. Consider reformulating the problem that is at the base of the given Diophantine equation/system such that a solution can be obtained. For instance, we may impose certain limits and restrictions on the domain of the problem which enable us to obtain full solution.
5. Consider obtaining partial (or tentative or conditional or ...) solutions when full (or certain or unconditional or ...) solutions could not be reached. Such "interim" solutions can form a basis for future attempts and investigations to obtain final solutions.[27]

[27] This recommendation (and the previous one) takes into consideration practical factors and issues. In brief, we should not give up! Moreover, we should always try to make use of what we could obtain and achieve (e.g. partial or conditional solution). This helps to maintain confidence and high spirit which help us in our future investigations of Diophantine problems. Such psychological factors are very important for our success in solving Diophantine problems. In fact, Diophantine equations is not an easy subject and hence without such confidence and high spirit our ability to solve these equations will be reduced and diminished.

15 Recommendations for Systems of Equations

Systems of Diophantine equations can be classified into two main categories: linear (when all the equations of the system are linear) and non-linear (when some or all of the equations of the system are non-linear). These two categories will be investigated in the following two subsections. However, before we go through this investigation we would like to draw the attention to the following points:

1. Many of the (previously discussed) recommendations related to individual Diophantine equations applies equally or similarly to systems of Diophantine equations as well (at least by considering the individual equations as part of the system but not the system itself). In fact, we already indicated (frequently) such system recommendations (at least within the given examples for certain recommendations and guidelines).
2. We should remind the reader of what we have said before (see § 1) that is: there are two main methods for solving systems of Diophantine equations in number theory. The first is based on using the traditional methods of solving systems of multivariate equations (as investigated in algebra and linear algebra for instance) such as by substitution or comparison or use of the techniques of matrices, and the second is by solving the individual equations separately (either by the general methods of algebra or by the special methods and techniques of number theory) and selecting the solutions that satisfy the system as a whole (i.e. by accepting only the solutions which are common to all the equations).

15.1 Systems of Linear Equations

Systems of linear Diophantine equations can be easily solved by the well known (and standard) methods of linear algebra where only the integer (or sub-integer such as positive integer) solutions are accepted. Other methods (including the methods used for solving systems of non-linear equations) are also possible to use in general for solving systems of linear equations (as will be discussed later).

15.2 Systems of Non-Linear Equations

Non-linear systems[28] of Diophantine equations can be solved usually by the well known algebraic methods of solving systems of equations (which are not restricted to Diophantine equations) such as by substitution, elimination and comparison (see § 15.2.4) where only integer (or sub-integer) solutions are accepted. However, it should be noted that certain methods (which are mostly investigated in linear algebra) are not applicable in solving non-linear Diophantine systems since these methods are specific to linear systems. In the following sub-subsections we will investigate briefly some of the common recommendations and guidelines (as well as methods and techniques) for solving non-linear systems of Diophantine equations. However, before that we would like to draw the attention to the following useful remarks:

1. It is recommended to test all the obtained (integer) solutions on the original system of equations. This is because some of the algebraic manipulations which are required during the process of solving systems of non-linear equations (such as raising to powers or multiplication or division by a variable) can introduce foreign solutions and hence by testing the obtained solutions we make sure that no foreign solution is introduced during these manipulations.
2. Systems of linear equations can be considered as a special type of systems of non-linear equations and hence they can be solved by the methods of systems of non-linear equations (when applicable) as well as by the methods and techniques which are specific to systems of linear equations (such as by the methods of matrices of linear algebra).
3. Unlike systems of linear Diophantine equations, there is no standard (or systematically applicable) method or technique for solving systems of non-linear Diophantine equations and hence solving these systems is a mix of art and science (where the art is required for selecting the main approach or strategy for solving the system while the science is needed for employing and applying the technicalities required by the selected approach).[29] Therefore, it is recommended to investigate various potential methods or strategies before setting off and making the choice of the best (or even applicable) strategy to use. Investing some time and effort in this initial investigation can be very rewarding and beneficial and can save considerable amount of time and effort in trying to solve the system in a rather random approach and chaotic manner.

[28] "Non-linear systems" should be more accurate than "systems of non-linear equations" if we consider systems that consist of some linear and some non-linear equations (which should be classified as non-linear systems although some of their equations are linear).

[29] In fact, even solving individual Diophantine equations is mostly a mix of art and science.

15.2.1 Initial Sensibility Checks

As indicated in § 2.6, initial sensibility checks are recommended (and even required) as the first step in tackling and solving Diophantine systems as well as in tackling and solving individual Diophantine equations. In many cases, solving the given system of equations does not need more than these initial sensibility checks since these checks can reveal that the system either has no solution (e.g. because one of the equations has no solution due to parity violation or modularity inconsistency or because some of the equations require conditions that contradict the conditions required by the other equations) or because the solution becomes so obvious by these initial sensibility checks.

For example, initial sensibility checks should reveal that the following system (where $x, y \in \mathbb{Z}$):

$$3xy + x^2 - 5y = 33 \qquad \text{and} \qquad 5x^3 + 10xy + 11y^2 = 23 \qquad \text{and} \qquad 17x + 4xy^2 = 147$$

has no solution because the second equation has no solution (noting that in modulo 5 this equation becomes $y^2 \overset{5}{=} 3$ which has no solution since 3 is not a quadratic residue of 5).

On the other hand, initial sensibility checks (or inspection) should reveal that the following system:

$$7^x - 8^y = 48 \qquad \text{and} \qquad 3x^2 - 5y^3 = 12 \qquad (x, y \in \mathbb{N}^0)$$

has (only) the obvious solution $(x, y) = (2, 0)$. This is because according to the first equation y must be 0 (to avoid parity violation) and hence $(x, y) = (2, 0)$ is the only possible solution to this system.

Also see the examples given in § 2.6.

15.2.2 Graphic Investigation and Reasoning

Some systems can be easily solved by graphic investigation and reasoning (which may or may not require plotting of actual graphs representing the equations of the system). For example, the following system of Diophantine equations:

$$x^2 - 2x + 4 + y = 0 \qquad \text{and} \qquad x^2 + y^2 + 6x - 10y + 30 = 0 \qquad (x, y \in \mathbb{Z})$$

was solved in § 3 by graphical reasoning without need for plotting any graph. Many other systems can be similarly solved either graphically (by plotting actual graphs) or by pure graphical reasoning (without plotting any graph). So, it is recommended to consider graphic investigation when tackling Diophantine systems (especially non-linear systems with two variables).

15.2.3 Test of Solutions of Known Equation

If the system of Diophantine equations contains an equation whose solution is known (or can be obtained easily or more easily) then the best approach for solving the system is to test the solutions of that equation on the other equations where only the common solutions (if any) to all equations are accepted. This is based on the obvious fact that the solution of the system is the intersection of the solutions of the individual equations (see the paragraph before the last of § 1) and hence the set of solutions of the system cannot exceed the set of solutions of any one of the equations in the system, i.e. the set of solutions of the system is a (proper or improper) subset of the set of solutions of any one of the equations in the system. This approach usually saves considerable amount of time and effort in trying to solve the system by other methods. In fact, in some cases this can be the only viable method for solving the system.

For example, the following system of Diophantine equations (where $x, y \in \mathbb{Z}$):

$$15x + 13xy - 20y = 0 \qquad \text{and} \qquad 3x^4 + 2y^3 + 202 = 0 \qquad \text{and} \qquad 7x^5 - 4y^2 - 9x^3y = 484$$

can be easily solved by testing the solutions of the first equation (assuming we have these solutions or they can be obtained rather easily) which are:

$$(x, y) = (2, -5),\ (0, 0),\ (-10, -1)$$

on the other equations in the system. By doing so we find out that only the solution $(x, y) = (2, -5)$ satisfies all the equations of the system and hence only this solution is acceptable as a solution to the system.

Similarly, the following system of Diophantine equations:

$$x^2 - 3y = 19 \qquad \text{and} \qquad 13y^3 + 6x = 11 \qquad \text{and} \qquad x^2 + y^2 = 17 \qquad (x, y \in \mathbb{Z})$$

can be easily solved by solving the last equation whose solutions[30] can be easily obtained (because there are only a few possibilities for x and y to satisfy this equation) and testing these solutions on the other two equations in the system. On doing so we find that only the solution $(x, y) = (4, -1)$ satisfies the other two equations and hence this is the only solution to the system.

15.2.4 Substitution, Elimination and Comparison

Substitution is one of the most common and straightforward methods for solving systems of Diophantine equations (whether linear or non-linear). For example, the following system:

$$3x^2 + 4y = 19 \qquad \text{and} \qquad 5x - 2y = 3. \qquad (x, y \in \mathbb{Z})$$

[30] The solutions of the last equation are: $(x, y) = (\pm 1, 4)$, $(\pm 1, -4)$, $(\pm 4, 1)$ and $(\pm 4, -1)$.

can be solved easily by substituting $2y = 5x - 3$ (which is obtained from the second equation) into the first equation to get $3x^2 + 10x - 25 = 0$ which is a univariate quadratic equation in x. Solving this quadratic equation (e.g. by the quadratic formula) will lead to the only solution of the system which is $(x, y) = (-5, -14)$.

Sometimes this method may require some minor (extra) manipulation prior to substitution (such as raising to power) to facilitate substitution. For example, the following system:

$$\sqrt{x} - 11y^2 = 8 \qquad \text{and} \qquad 4xy - 1033y^2 = 411$$

can be solved by substituting $x = (11y^2 + 8)^2$ (which is obtained from the first equation by squaring $\sqrt{x}$) into the second equation to get $484y^5 + 704y^3 - 1033y^2 + 256y = 411$ whose only integer solution is $y = 1$ (i.e. $x = 19^2 = 361$) and hence the only solution to the given system is $(x, y) = (361, 1)$.

The methods of elimination and comparison are very similar to the method of substitution (in their technicalities as well as in their wide applicability) and hence we do not need to investigate them.

16 Testing the Final Solutions

It is strongly recommended to test the obtained solutions (whether of individual equations or of systems of equations) on the given equations and systems, e.g. by substituting the values of the variables of these solutions in the given equations and systems. This is particularly important in the following cases:

• When the solutions are obtained with certain manipulations that can introduce foreign solutions (see § 4).

• When the solutions are obtained by messy arguments and formulations and hence it is likely that the solutions are wrong or they contain errors and mistakes.

We may also consider using computational tools to do final checks, e.g. by running a code to obtain the solutions (as we did in § 5) and compare them to the already obtained solutions. In fact, we strongly recommend using such computational tools (like coding or spreadsheets or software packages) to check the final answer especially when we have some doubt or when the produced argument (or proof or formulation or ...) is very messy and susceptible to errors and mistakes. So, we recommend *starting* our investigation of Diophantine problems by computational inspection to probe the solution and form a general idea about it (see § 5), and *terminating* our investigation by computational testing and checking of the obtained solution.

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Nomenclature

In the following list, we define the common symbols, notations and abbreviations which are used in the paper as a quick reference for the reader.

$!$	factorial
$\in$	in (or belong to)
$\ni$	(backward) in (or belong to)
$\|a\|$	absolute value of a
Eq., Eqs.	Equation, Equations
LHS, RHS	left hand side, right hand side
$m \overset{k}{=} n$	m and n are congruent modulo k
$m \overset{k}{\neq} n$	m and n are not congruent modulo k
mod	modulo (or modulus)
$\mathbb{N}$	the set of natural numbers (i.e. $1, 2, 3, \ldots$)
$\mathbb{N}^0$	the set of non-negative integers (i.e. $0, 1, 2, 3, \ldots$)
$O_k n$	the order of integer n (modulo k)
p	prime number
$\mathbb{P}$	the set of prime numbers
$\mathbb{Z}$	the set of integers
Δ	discriminant of quadratic polynomial

Author Notes

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