EXPLANATIONS OF THE RIEMANN HYPOTHESIS

DMITRI MARTILA

ABSTRACT. Explanations why the real part of Zeta function zeroes is always being seen on the 1/2 line.
MSC Class: 11M26, 11M06.

1. First explanation

There is a vivid interest in the Riemann Hypothesis, while there are no reasons to cast doubt on the validity of the Riemann Hypothesis [1]. This hypothesis was proposed by Bernhard Riemann (1859). Many colleagues consider it the most important unsolved problem in pure mathematics [2]. The Riemann Hypothesis is of great interest in number theory because it implies results about the distribution of prime numbers. Our first contribution to the field is available from arXiv [3]. Because it is not refuted, we regard it as the first explanation.

2. Second explanation

Already from the Riemann's paper, we know that if a zero of the zeta function $\zeta(s)$ does not belong to the critical line $s = 1/2 + i\,y$, there are two zeroes which are symmetric about this line: $s_1 = 1/2 + v + i\,y$, $s_2 = 1/2 - v + i\,y$, for $0 < v < 1/2$. One has $\zeta(s_1) = \zeta(s_2) = 0$. Let us multiply the zeta function by some function $A(s)$ to obtain $\beta(s) = A(s)\zeta(s)$. Then, again, one has $\beta(s_1) = \beta(s_2) = 0$.

Hence, the system for finding zeroes of the zeta function is given by $\zeta(s_1) = \zeta(s_2)$ and $\beta(s_1) = \beta(s_2)$. This system is satisfied at the critical line $s = 1/2 + i\,y$, where $s_1 = s_2$.

So, let us look closer at our system. The $\zeta(s_1) = \zeta(s_2)$ is giving

$$\Re\,\zeta(s_1) = \Re\,\zeta(s_2)\,, \tag{1}$$

$$\Im\,\zeta(s_1) = \Im\,\zeta(s_2)\,. \tag{2}$$

The $\beta(s_1) = \beta(s_2)$ is giving

$$A(s_1)\zeta(s_1) = A(s_2)\zeta(s_2)\,. \tag{3}$$

---

eestidima@gmail.com, Independent Researcher
J. V. Jannseni 6–7, Pärnu 80032, Estonia.

Applying Eqs. (1), (2), we come to

$$(A(s_1) - A(s_2))\zeta(s_1) = 0\,. \tag{4}$$

In conclusion, the system $\zeta(s_1) = \zeta(s_2)$ and $\beta(s_1) = \beta(s_2)$ with $s_1 \neq s_2$ results in $\zeta(s_1) = 0$.

Is known Landau's xi function $\xi = A\,\zeta$, where specific A makes $\beta(s_1) = \beta(s_2)$ to hold automatically because $\xi(s) = \xi(1-s)$ is the functional equation; and the complex-conjugate of it has $\xi(s^*) = \xi^*(s) = \xi^*(1-s) = \xi(1-s^*)$. This means that the system of just two equations Eqs. (1), (2) is enough for searching for the zeroes.

Now, we can start with the Zeta function, multiply it by arbitrary $B(s)$, $q = B\,\zeta$ and come the above way to the conclusion, that $q(s_1) = q(s_2)$ gives zeroes with $s_1 \neq s_2$ instead of $\zeta(s_1) = \zeta(s_2)$. We come to a contradiction.

3. Third explanation

The number of zeroes in the critical strip $0 < \Re(s) < 1$ within the range $0 < \Im(s) \leq T$ for the imaginary part is given by [4]

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1/T)\,, \tag{5}$$

where

$$S(T) = \pi^{-1}\,\mathrm{Arg}\,\zeta(1/2 + i\,T)\,. \tag{6}$$

Hence, all jumps (a jump is a discontinuity of a function) in the amount of zeroes $N(T)$ happen due to $S(T)$. But $S(T)$ belongs to the critical line, where only one zero is possible. Hence, there is only one zero per T. It is on the 1/2 critical line.

References

[1] David W. Farmer, "Currently there are no reasons to doubt the Riemann Hypothesis," arXiv:2211.11671 [math.NT], 2022AD.

[2] Enrico Bombieri, (2000), The Riemann Hypothesis – official problem description, Clay Mathematics Institute.

[3] Dmitri Martila, Stefan Groote, Riemann's hypothesis via Robin's Theorem, arXiv:2401.07635 [math.NT], https://doi.org/10.48550/arXiv.2401.07635

[4] E. C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2007, page 212; Timothy S. Trudgian, "An improved upper bound for the argument of the Riemann zeta function on the critical line II." J. Number Theory, 134: 280–292 (2014). https://doi.org/10.1016/j.jnt.2013.07.017 R. J. Backlund, Über die Nullstellen der Riemannschen Zetafunktion. Acta Math. 41, 345–375 (1916). https://doi.org/10.1007/BF02422950