Improper integrals of the second kind

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ABSTRACT: This document briefly discusses improper integrals of the second kind.

I. Introduction

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of the second kind.

II. Definition

1. If \( f(x) \) is continuous on \((a,b] \) and is discontinuous at \( a \) then

\[
\int_{a}^{b} f(x) \, dx = \lim_{c \to a^+} \int_{c}^{b} f(x) \, dx
\]

(1)

2. If \( f(x) \) is continuous on \([a,b) \) and is discontinuous at \( b \) then

\[
\int_{a}^{b} f(x) \, dx = \lim_{c \to b^-} \int_{a}^{c} f(x) \, dx
\]

(2)

3. If \( f(x) \) is discontinuous at \( c \), where \( a < c < b \), and continuous on \([a, c) \cup (c, b] \) , then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

(3)

In each case, if the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges.

III. Special improper integrals of the second kind

1. \[
\int_{a}^{b} \frac{1}{(x-a)^p} \, dx \text{ converges if } p < 1 \text{ and diverges if } p \geq 1
\]

(4)

2. \[
\int_{a}^{b} \frac{1}{(b-x)^p} \, dx \text{ converges if } p < 1 \text{ and diverges if } p \geq 1
\]

(5)

These can be called \( p \) integrals of the second kind. Note that when \( p \leq 0 \) the integrals are proper.

IV. Convergence tests

The following tests are given for the case where \( f(x) \) is unbounded only at \( x = a \) in the interval \( a \leq x \leq b \). Similar tests are available if \( f(x) \) is unbounded at \( x = b \) or \( x = c \) where \( a < c < b \).

1. Comparison test for integrals with non-negative integrands.
1.1. Convergence. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x) \, dx$ converges. Then if $0 \leq f(x) \leq g(x)$ for $a < x \leq b$, $\int_a^b f(x) \, dx$ also converges.

1.2. Divergence. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x) \, dx$ diverges. Then if $f(x) \geq g(x)$ for $a < x \leq b$, $\int_a^b f(x) \, dx$ also diverges.

2. Quotient test for integrals with non-negative integrands.

2.1. If $f(x) \geq 0$ and $g(x) \geq 0$ for $a < x \leq b$, and if $\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = A \neq 0$ or $\infty$, then $\int_a^b f(x) \, dx$ and $\int_a^b g(x) \, dx$ either both converge or both diverge.

2.2. If $A = 0$ in 2.1, and $\int_a^b g(x) \, dx$ converges, then $\int_a^b f(x) \, dx$ converges.

2.3. If $A = \infty$ in 2.1, and $\int_a^b g(x) \, dx$ diverges, then $\int_a^b f(x) \, dx$ diverges.

V. Examples

1. Example 1.

$$I = \int_0^1 \frac{2}{\sqrt{1-x^2}} \, dx$$

$$a = 0, \ b = 1, \ f(x) = \frac{2}{\sqrt{1-x^2}}, \ f(1) \to \infty, \ g(x) = \frac{1}{\sqrt{1-x}}, \ \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{2}{\sqrt{1+x}} = \sqrt{2}$$

$$\int_0^1 g(x) \, dx = \sqrt{2}$$

(7) and (8) imply $\int_0^1 \frac{2}{\sqrt{1-x^2}} \, dx$ converges

2. Example 2.

$$I = \int_0^{\sqrt{2}} f(x) \, dx$$

where

$$f(x) = \sqrt{1 + \frac{1}{2} \sqrt{3} \left( \frac{4}{x^2} + \frac{16 \cdot 2^{1/3}}{h(x)} + \frac{h(x)}{2^{1/3} x^4} + \frac{1}{2} \left( \frac{f(x) - 16 \cdot 2^{1/3}}{3 h(x)} - \frac{h(x)}{3 \cdot 2^{1/3} x^4} + \frac{\sqrt{3}}{4} \right) s(x) \right)$$

$$h(x) = \left( 27 x^4 + 128 x^6 + \sqrt{729 x^8 + 6912 x^{10}} \right)^{1/3}$$

$$t(x) = 8 - \frac{2 \left( -1 + 3 x^2 \right)}{x^2} - \frac{2 \left( -x^2 + 3 x^4 \right)}{3 x^4}$$

$$s(x) = -64 + \frac{32 \left( -1 + 3 x^2 \right)}{x^2} - \frac{8 \left( -1 + 4 x^2 + 4 x^4 \right)}{x^4}$$

$$a = 0, \ b = \sqrt{2}, \ f(0) \to \infty, \ g(x) = \frac{1}{x^{3/4}}, \ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x^{3/4} f(x) = 0$$
\[
\int_0^{\sqrt{2}} g(x) \, dx = 4 \cdot 2^{1/8}
\]

(16)

\[
(15) \land (16) \implies \int_0^{\sqrt{2}} f(x) \, dx \text{ converges}
\]

(17)

Recall that

\[
\pi = \int_0^1 \frac{2}{\sqrt{1 - x^2}} \, dx
\]

(18)

\[
\pi = \int_0^{\sqrt{2}} \left[-1 + \frac{1}{2 \sqrt{3}} \sqrt{\frac{4 + 16 \cdot 2^{1/3}}{x^2} + \frac{h(x)}{2^{1/3} x^4}} + \frac{1}{2} \sqrt{\frac{r(x) - 16 \cdot 2^{1/3}}{3 h(x)}} - \frac{h(x)}{3 \cdot 2^{1/3} x^4} + \frac{\sqrt{3} \cdot s(x)}{4 \sqrt{\frac{5}{x^6} + 16 \cdot 2^{1/3} h(x)}} + \frac{h(x)}{2^{1/3} x^4}} \, dx
\]

(19)

\[
\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)
\]

(20)

VI. References


