ABOUT COUNTABILITY SETS

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Abstract

A refutation of Cantor's diagonal argument. A proof of the equality of the number of elements in the sets of natural and real numbers. A refutation of Cantor's power set theorem for infinite sets. A proof of the equality of the number of elements in an infinite set and its power set. Geometric proof. The relationship between infinity and the number zero. The theorem on the countability of all sets.

INTRODUCTION

At the end of the 19th century, German mathematician Georg Cantor published a paper proving that one infinite set could be larger than another infinite set. This theory sparked considerable controversy among mathematicians from the very beginning. One of its proponents, David Hilbert, once said: "Cantor has taken us into the endless world of infinity, and from this paradise no one will drive us away." But there were also mathematicians like Leopold Kronecker, who considered it more of a mathematical hell than a paradise. In 1878, Cantor proposed his famous continuum hypothesis[1], based on power sets of infinite sets. In the 1940s and 1960s, Kurt Gödel[2] and Paul Cohen[3] proved this hypothesis independence from the axioms of ZFC. Because of this, these infinities could be discrete like natural numbers on the number line, as Cantor conjectured, but they could just as easily be continuous like real numbers. Perhaps their density would be even greater. We would thus have an undefined mess of infinities, with no way to determine its structure within the generally accepted axioms of ZFC. One can therefore understand the reluctance of some mathematicians to accept this theory. Infinity is a difficult abstraction to understand, and if there were an infinite number of them, it would become infinitely difficult. If Cantor were right, this world of infinity would be not only undefined but also incomprehensible. It is therefore worth considering whether it exists at all.

Cantor’s diagonal argument[4]

Cantor's argument is presented as follows. Suppose that all real numbers in the interval (0,1) are arranged in a sequence. For example, this one:

$$\begin{array}{c}
1 \leftrightarrow 0.\mathbf{4}249015\ldots \\
2 \leftrightarrow 0.1\mathbf{9}39572\ldots \\
3 \leftrightarrow 0.89\mathbf{2}7446\ldots \\
4 \leftrightarrow 0.553\mathbf{6}898\ldots \\
5 \leftrightarrow 0.3499\mathbf{5}02\ldots \\
\ldots \\
\ldots
\end{array}$$

On the diagonal we have a certain real number from the interval (0,1) marked in bold:

$$0.49265\ldots \tag{1.1}$$

This number has the same digit in the first decimal place as the first number in the sequence. At the second decimal place is the same digit as the second digit in the second number in the sequence. Similarly, each subsequent number in the sequence has one digit in common with number 1.1. To each digit in this number, we can add one; if the digit is nine, we enter a zero. This produces a new number:

$$0.50376\ldots \tag{1.2}$$

Cantor claims that the number 1.2 differs from every number in the sequence by at least one digit and concludes from this that this number is not in the sequence, which means that it is always possible to construct a real number that is not in the sequence. According to Cantor, this proves the uncountability of real numbers.

Note, however, that a number constructed in this way cannot be any number in the interval (0,1) because, according to the assumption, all these numbers are in the sequence, and the number constructed by Cantor is different from each of them. Exists many real numbers that don't belong to this interval, but they don't matter to us. We're only interested in numbers in the interval (0,1) because we want to determine whether the set of these numbers is countable or not. This number cannot be a proof of the uncountability of real numbers. At the same time, this number is constructed in such a way that it should belong to the interval $(0,1) \ni l = 0.a_1a_2a_3\ldots$ This constitutes a contradiction. Such a number does not exist because it would have to both belong and not belong to the interval (0,1). Cantor claims that we can always construct a number that is not in this sequence, but as we can see, this is not the case. Cantor claims that the fact that all numbers are in the sequence implies that not all numbers are in the sequence. This argument is flawed and proves nothing. Indirect proof relies on a contradiction, but this contradiction cannot be a denial of the truth of an assumption. Because if it contradicts the assumption on which we based our inference, then our inference is also false. Thus, we fall into a paradoxical loop similar to the liar's paradox. Diagonalization arguments are often also the source of contradictions like Russell's [5] paradox and Richard's[6] paradox. At this stage of our considerations, we still don't know whether the set of real numbers is countable. Even if the set $\mathbb{R}$ were uncountable, this would require another way of proving it. Let's look at it another way.

COUNTABILITY OF REAL NUMBERS

Let's begin by reviewing the proof of the countability of rational numbers. We can demonstrate this using a table containing all positive rational numbers and the number zero, and a similar table for negative rational numbers.

	0	1	2	3	4	…
1	$\frac{0}{1}$↙	$\frac{1}{1}$↙	$\frac{2}{1}$↙	$\frac{3}{1}$↙	$\frac{4}{1}$↙	…
2	$\frac{0}{2}$↙	$\frac{1}{2}$↙	$\frac{2}{2}$↙	$\frac{3}{2}$↙	$\frac{4}{2}$↙	…
3	$\frac{0}{3}$↙	$\frac{1}{3}$↙	$\frac{2}{3}$↙	$\frac{3}{3}$↙	$\frac{4}{3}$↙	…
4	$\frac{0}{4}$↙	$\frac{1}{4}$↙	$\frac{2}{4}$↙	$\frac{3}{4}$↙	$\frac{4}{4}$↙	…
…	…	…	…	…	…	…

	-1	-2	-3	-4	…
1	$\frac{-1}{1}$↙	$\frac{-2}{1}$↙	$\frac{-3}{1}$↙	$\frac{-4}{1}$↙	…
2	$\frac{-1}{2}$↙	$\frac{-2}{2}$↙	$\frac{-3}{2}$↙	$\frac{-4}{2}$↙	…
3	$\frac{-1}{3}$↙	$\frac{-2}{3}$↙	$\frac{-3}{3}$↙	$\frac{-4}{3}$↙	…
4	$\frac{-1}{4}$↙	$\frac{-2}{4}$↙	$\frac{-3}{4}$↙	$\frac{-4}{4}$↙	…
…	…	…	…	…	…

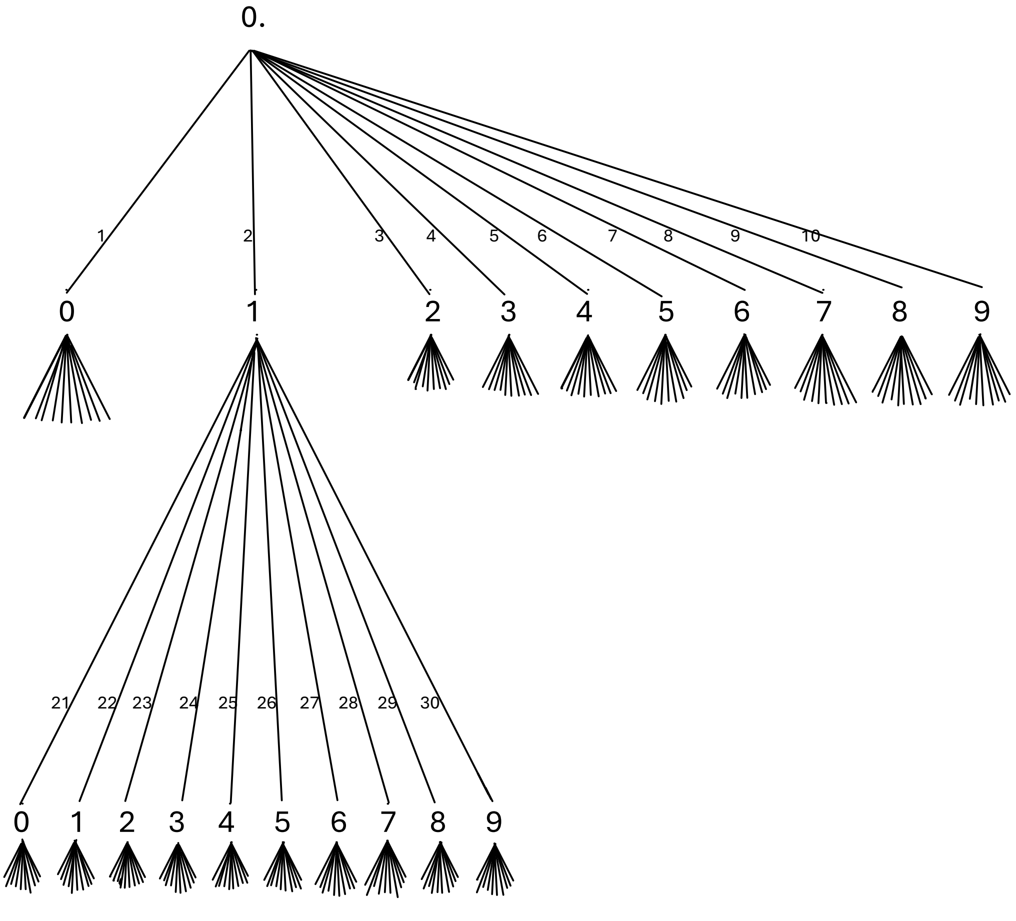
Now we can number them all diagonally from the upper right to the lower left. We can assign even natural numbers to positive rational numbers, and odd natural numbers to negative rational numbers. We omit repeating rational numbers.

$$1 \leftrightarrow -\frac{1}{1}$$
$$2 \leftrightarrow \frac{0}{1}$$
$$3 \leftrightarrow -\frac{2}{1}$$
$$4 \leftrightarrow \frac{1}{1}$$
$$5 \leftrightarrow -\frac{1}{2}$$
$$6 \leftrightarrow \frac{2}{1}$$
$$7 \leftrightarrow -\frac{3}{1}$$
$$8 \leftrightarrow \frac{1}{2}$$
$$\dots$$

This gives us a bijection of natural numbers and rational numbers. Two things are crucial here: that these tables potentially contain all rational numbers, and that each diagonal contains a finite quantity of rational numbers. Thanks to this, we can number all rational numbers. This proves the countability of rational numbers.

REAL NUMBERS

Let us try to show the countability of real numbers in an analogous way. Each number in this range has the form $0.a_1a_2a_3\dots$ As we can see, the quantity of digits is countable. There may be at each place after the comma ten different digits. Let’s create a tree that represents the numbers of the numerical interval $[0\,,1)$. We start with a zero to which we assign ten digits, and then to each of them another ten digits, and so on.



Note that every real number is more than just a sequence of digits. For example, the irrational number $\pi - 3 = 0.1415 \ldots$ has all its digits predetermined, even though there are an infinite number of them. Each digit is defined by the ratio of the circumference of a circle to its radius. At infinity, all digits are fixed and immutable. This number is represented by one of the branches in our tree, and only one. We can therefore choose a specific branch and assign it a natural number. Similarly, every real number in its decimal representation in the interval [0,1) has a corresponding branch. Two important things here are that this tree contains all the real numbers in the interval [0,1) and that each level has a finite number of branches. We can number all the branches, and none will be omitted. To each natural number, we assign another infinite branch of our choice, representing a real number. By numbering all the branches consecutively at each level, we establish a bijection between the set of natural numbers and the set of real numbers. Georg Cantor proved that every nonempty interval of real numbers is equivalent to the entire set ℝ. This proves the countability of the set of real numbers. We can arrange all real numbers in the interval [0,1) into a sequence.

$$\begin{gathered} 1 \leftrightarrow 0.012851 \ldots \\ 2 \leftrightarrow 0.141592 \ldots \\ 3 \leftrightarrow 0.285704 \ldots \\ 4 \leftrightarrow 0.308887 \ldots \\ \ldots \\ 25 \leftrightarrow 0.140592 \ldots \\ \ldots \end{gathered}$$

There are some digits on the diagonal. If we add number one to each of them, we get a number that isn't in the sequence. This is a contradiction, because all the numbers, as we can see, are in the sequence. This means that the digits on the diagonal don't form a number either. It's a sequence of chaotic digits. As we noted earlier, each number has all its digits predetermined. The digits on the diagonal are not ordered, so their order cannot be determined in advance. This explains why we can't construct a number that isn't in the sequence. If the numbers in the sequence were ordered, then one could deduce the order of the digits on the diagonal, which means a contradiction. The numbers in the sequence will also appear chaotically. We can rank all the numbers in the interval (0,1), but we can only do this by selecting them randomly one after the other. We can arrange them all, but we can only do so by randomly selecting them one by one. There's no formula that specifies how to do this. In this tree, there's no single branch we could choose that would represent the sequence of all the digits on the diagonal. Therefore, it's impossible to assign it a natural number and arrange it into a sequence. For a finite number of digits on the diagonal, there's only a part of a branch who representing them, but not for all the digits.

We have a few unusual things here. The first is potential infinity. The next is a sequence of digits that don't form a number. Another is potential chaos; if we have all infinitely many digits, they're not in any way ordered. Also, numbers in the sequence to infinity are potentially chaotic. The next thing that appears here is a bijection between the set of natural numbers and the set of real numbers in the interval [0,1), which can't be described by a formula. In such a bijection, like y = x, this formula describes the Cartesian product ℝ × ℝ. In the case of our sequence, there's no formula that describes it still, it is a function of $\mathbb{N} \times [0,1)$. Even more strange is that on the left side of the sequence we have the order of the natural numbers and on the right side the chaos of the real numbers.

CANTOR'S THEOREM for power set [7]

Each set has less power than the family of its subsets, i.e. power set.

Proof:
Let $f: A \to P(A)$ be any function from a given set A into its power set $P(A)$. Let's define a set B of those elements of set A that do not belong to their images in the function f :

$$B = \{x \in A: x \notin f(x)\}$$

Set B , as a subset of set A, is an element of power set A:

$$B \subseteq A \quad \Longrightarrow \quad B \in P(A)$$

Therefore, for any element m belonging to set A , there is:

$$m \notin f(m) \ \Longrightarrow m \notin f(m) \wedge m \in B \Longrightarrow f(m) \neq B \qquad (2.1)$$

$$m \in f(m) \ \Longrightarrow m \in f(m) \wedge m \notin B \Longrightarrow f(m) \neq B \qquad (2.2)$$

Thus, the set B is not an image of any element of the set A in the mapping f, hence the function f cannot be a surjection (the "onto" function), and in particular cannot be a bijection. This means that the sets A and $P(A)$ are not equal $|A| \neq |P(A)|$.

The proof of the theorem seems convincing, but only apparently.
It seems that set B denies the existence of the bijection of the set into the power set. In fact, the opposite is true, it is bijection that prevents the formation of set B. It is the function f that creates the set B and does not vice versa. When the function f is a bijection, then for every subset of set A there is some element for which this subset is the image. That is, for the subset B there is some element m such that $f(m) = B$. According to the definition of the set B it is not the image of its elements, i.e. $m \notin B$. However, this means that m satisfies the condition $x \notin f(x)$, i.e. it is an element of set B, therefore:
If f is bijection then

$$\exists m \in A \quad f(m) = B \Longrightarrow m \notin B \wedge m \in B$$

that means a contradiction. The set B does not exist, the bijection f does not define such a set. Illustratively, when the function f assigns an element m to a subset of B_n containing only elements satisfying the predicate $m \notin f(m)$, it simultaneously adding another element to it. This function works in such a way that when we want to create a set B, it already has an element m from set A assigned to it, which is not an element of set B. This means that we have to include it in set B. This way we obtain a new set B, which already has another element from set A assigned to it, and again we have to create a new set B. Since the function works on infinite sets, the set will never be created; it is created endlessly. This is a version of Russell's paradox. There would have to be an element that both belongs and does not belong to this set. For example, we could try to create the largest set that would contain all sets except itself $M = \{A_1, A_2, A_3, \dots\}$. But then we could create a new set that contains all the previous sets A and an additional element, which is the set M. We obtain a new set $M_1 = \{M, A_1, A_2, A_3, \dots\}$ of which M is a proper subset. In this way, we construct subsequent sets that contain the previous ones $M \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$ and we can do this indefinitely. Therefore, there is no greatest set, i.e., the set of all sets. The same applies to our set B.
Can such a bijection exist? We can show that yes.

Assume that the function f is not a bijection and (2.1) and (2.2) hold. There are subsets Bn containing some (but not all) elements from the set A satisfying the predicate $m \notin f(m)$. Is there the set of B in which for every n Bn$\subset B$? For this function, yes. The set B is not the image of any element of the set A in the mapping f. Let's assume that all subsets that are elements $f(A)$ are images of some element from set A except B. We can create a new function such that:

$$g(a_n) = \begin{cases} B, & n = 1 \\ f(a_{n-1}), & n > 1 \end{cases}$$

Such a function is a bijection of sets A and $P(A)$. For the function g there are different sets of B_n than for function f, but for each, there is an element from the set A for which B_n is an image. That is, there is no set of B for the function g. The predicate $x \notin g(x)$ determines a class of sets [8], not a set.

$$\forall i \exists j \ B_i \subset B_j$$

The series of sets of B_n determined by bijections is infinite and divergent, and there is no sum.

For finite sets it is impossible to construct the function g, i.e. the bijection for sets A and $P(A)$ can only exist for sets having infinitely many elements. For finite sets, Cantor's theorem is true. Set B may exist for some functions, such as non-bijection and only such.

We have shown that if there is a bijection between an infinite set and its power set, then the proof of Cantor's theorem is flawed. We have also shown that such a function can exist. Now we will show that it does exist.

Let's consider the power set of the infinite set A. In the set $P(A)$ is two types of sets, finite sets and infinite sets. Let's designate them as $P`(A)$ and $P``(A)$ respectively. Finite sets are a countability many.

$$|P`(A)| = |A^1| + |A^2| + |A^3| + \cdots = \aleph_0 + \aleph_0 + \aleph_0 + \cdots = \aleph_0$$

We can notice that:

$$\begin{gathered} \forall P`` \in P``(A) \ \ \exists a_1, a_2, a_3, \ldots \in A \ \ P`` = \{a_1, a_2, a_3, \ldots\} \Longrightarrow \\ \Longrightarrow \ \forall a_n \in P`` \ \exists P`_n \in P`(A) \ \ P`_n = \{a_1, a_2, \ldots, a_n\} \end{gathered}$$

That is, for any finite number of elements of an infinite subset, there is a finite subset containing these elements and only these. This is a very important observation. This means that infinite sets cannot be more than finite sets. Intuitive is like there are as many singletons as there are elements in a given set. Otherwise, there would have to be the largest finite set. Then the infinite subset would have to consist of several finite subsets, meaning there would be more combinations, so that would be more infinite subsets than finite subsets. There is no largest finite subset. Therefore, there are no more infinite subsets than finite subsets. We can therefore arrange infinite subsets by means of finite subsets by arranging them in pairs. First, we can rank subset $P`(A)$:

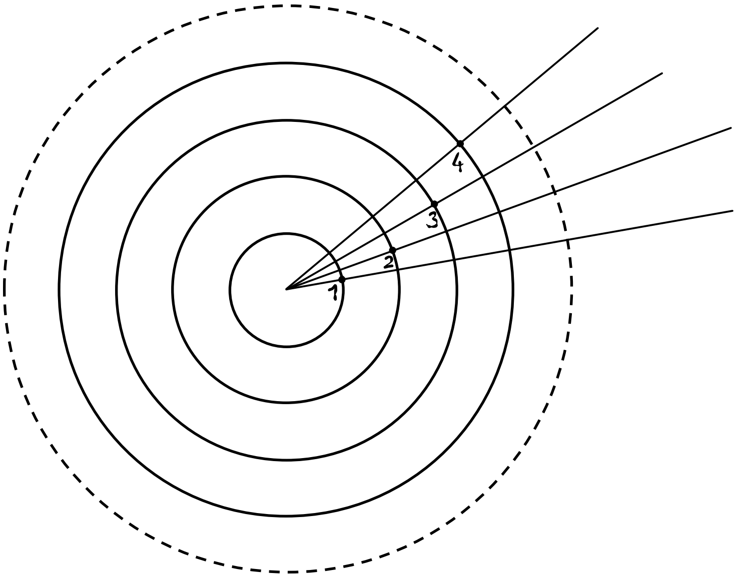
$$P`(A) = \{P`_1, P`_2, P`_3, \ldots\}$$

We can use finite subsets to rank infinite subsets. Let us assign finite subsets to infinite subsets such that $P`_n \subset P``$ . This way we assign a natural number n to a subset $P``$. Let us impose an additional condition $P`` \neq P``_1, P``_2, \ldots, P``_{n-1}$. Such an assignment is one-to-one. For every $P`` \in P``(A)$ and $P`` \neq P``_1, \ldots, P``_n$ , there exists another $P`_k$ where $n < k$ such that $P`_k \subset P``$. Therefore, this is a function. For every $P`_n$, there exists a $P``$ containing it. This is "onto" function. This is how we defined the bijection $f: P`` \to P`$. This proves that in a power set there is a countable number of infinite subsets. Therefore, there exists a function g

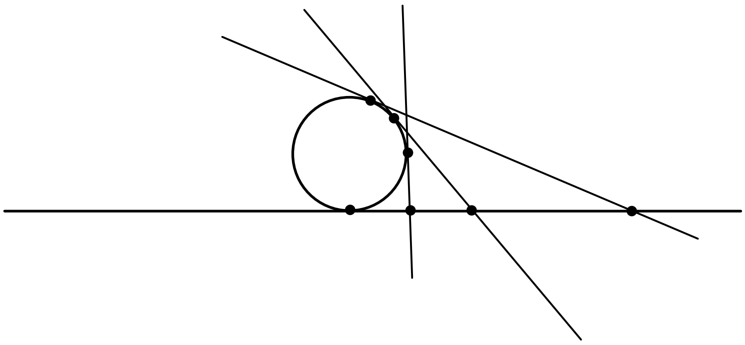
$$g(n) = \begin{cases} P`_n \in P`(A), & 2n-1 \\ P``_n \in P``(A), & 2n \end{cases}$$

We can see an apparent paradox here. For finite sets, there are only finite subsets, and there are more of them than elements of the set, which is expressed by the formula $|P(A_n)| = 2^n$. On the other hand, for infinite sets, in addition to finite subsets, we have infinite subsets and there is no more of them as the elements of the set. This is because there is no greater quantity than infinitely many. Just as it may seem that there are more $\mathbb{Z}$ numbers than $\mathbb{N}$, but at infinity there is no more number of them. Similarly, is no more real numbers then natural numbers.

We can demonstrate the countability of real numbers geometrically. As noticed earlier, every circle has the same number of points. This is proven by drawing a ray from the centres of two concentric circles that has one point in common with each circle. This way, we pair all the points on both circles. Different points on one circle will correspond to different points on the other circle. Through each point, we can draw such a ray. This shows that the number of points does not increase with increasing radius. Imagine an infinite number of concentric circles, each with a radius greater than the previous one by a fixed unit.



Amount circles are countability. On the first circle, we assign the natural number one to the point of intersection with the first ray. We draw another ray, and to the point of intersection with the second circle, we assign another natural number two. We proceed in the same way with the subsequent circles. By assigning a natural number to a point on the circle, we simultaneously assign the same number to the corresponding points on all circles at the point of intersection of the same ray. At infinity, all points on each circle will be assigned a natural number because a ray can be drawn through each point. If a point on the circle were not labelled with a number, it would be labelled on the next circle. Since each circle has the same number of points, no point will be omitted. Only if the number of points increased with its radius, we would not be certain that all points were assigned a natural number. In this way, we have numbered the points on the circle using circles. We can therefore say that countability is a property of infinity. We can connect points on the circle to points on the line using tangents.



We identify points on a line with real numbers. In this way, we have paired natural numbers with real numbers. This proves that there are no more real numbers than natural numbers.

In many mathematical equations, we can find such a fraction $\frac{1}{\infty}$. It is assumed that this fraction tends to zero. We write it like this

$$\frac{1}{\infty} \rightarrow 0$$

This means that this fraction is infinitely close to the number zero. This notation is a generalization of sequences that converge to zero. This shows that there is a relationship between infinity and the number zero. Let us assume that there are smaller and larger infinite sets and therefore smaller and larger cardinal numbers denoting infinity.

$$\begin{array}{ccc} \aleph_0 < \aleph_1 & \Rightarrow & \frac{1}{\aleph_0} > \frac{1}{\aleph_1} \\ & & \downarrow \quad\;\; \downarrow \\ \text{that is} & & 0_0 > 0_1 \end{array}$$

That means there would have to be smaller and larger numbers zero. The number zero is identified with the empty set. Within the axioms ZF there is only one empty set. There are no two different empty sets, so there are no smaller and larger empty sets. Similarly, there are no smaller and larger zero numbers. It follows that our assumption $\aleph_0 < \aleph_1$ is false. If, however, both fractions tended to the same number zero, then by would mean that they differ from each other by an infinitesimal amount, therefore these cardinal numbers at infinity would have to be equal.

$$\frac{1}{\infty} \rightarrow 0 \quad \Rightarrow \text{ only one: } \emptyset, 0, \infty$$

We can therefore make a more general theorem.

THEOREM

Each set is countability. Any two sets of infinitely many elements are of the same power.

$$\forall A \quad |A| \leq \aleph_0 \qquad\qquad \forall A_\infty \forall B_\infty \quad |A_\infty| = |B_\infty|$$

This solves the so-called continuum hypothesis problem. This hypothesis assumes that Cantor's theorem about the existence of smaller and larger infinities is true. Since it is not so then naturally the question of which infinity is the next after the infinity of natural numbers loses its meaning. There is no cardinal number continuum c or anything greater than aleph zero. Quantitatively, there is nothing more than infinity. Within the axioms of ZFC it is impossible to construct an uncountable set. Set means certain elements that we treat as a whole, but it does not determine where these elements are located or how they relate to each other. This means that we can "take" any element of the set and put it in the first place in the sequence. Then we can put any other element of the set in this series and so on. We don't need any formula for this; we can select elements completely chaotically. There are no mathematical rules to prevent us from doing so and there is no element that we cannot place in a sequence, so any set is countability. We can therefore say that the countability of a set is something very fundamental and results from the very nature of sets. For a quantity greater than infinity to exist, something other than a set would have to be invented, something to replace the concept of a set, and it would mean inventing alternative mathematics. This seems impossible.

Reference:

[1] Cantor, Georg (1878). "Ein Beitrag zur Mannigfaltigkeitslehre". Journal für die Reine und Angewandte Mathematik. 1878 (84): 242–258. doi:10.1515/crll.1878.84.242 (inactive 17 December 2025).

[2] Gödel, Kurt (1940). The Consistency of the Continuum-Hypothesis. Princeton University Press.

[3] Cohen, Paul J. (15 December 1963). "The independence of the Continuum Hypothesis, [part I]". Proceedings of the National Academy of Sciences of the United States of America. 50 (6): 1143–1148.

[4] Georg Cantor (1891). "Ueber eine elementare Frage der Mannigfaltigkeitslehre". Jahresbericht der Deutschen Mathematiker-Vereinigung. (1: 75-78) English translation: Ewald, William B. , ed. (1996).

[5] Bertrand Russell (1931). Principles of mathematics. Norton. pp. 363–366.

[6] Richard, Jules (1905). Les Principes des Mathématiques et le Problème des Ensembles. Revue Générale des Sciences Pures et Appliquées. Translated in Heijenoort, J. van, ed. (1964). Source Book in Mathematical Logic 1879-1931. Cambridge, MA: Harvard University Press.

[7] Thomas Jech (2002), "Set Theory" (The Third Millennium Edition, Revised and Expanded) Springer Monographs in Mathematics, Springer, p. 27-28 ISBN 3-540-44085-2

[8] John Kelley: General topology. 1976 (1955). ISBN 0-387-90125-6.