ABOUT COUNTABILITY SETS

Tomasz Soltysiak (Sołtysiak) May 2024

Abstract

Undermining Cantor's theorem about power sets for infinite sets. Proof of equal number elements of sets of natural and real numbers. Proof of equal number elements of an infinite set and its power set. Theorem about countability of all sets.

NUMBER OF CANTOR

Georg Cantor argued that there are more elements in a set of real numbers than there are in a set of natural numbers.[1]

A version of the proof looks like this:

Let's assume that all real numbers from the numerical interval [0 , 1) can be set in an infinite sequence, i. e. assign a natural number to each real number. Cantor claimed that we could construct a number that's not in this sequence. The way we do this is we take the first digit after the decimal point from the first number and add one, and from the second number, we take a second digit and add one again. We do the same with the next numbers, and if the digit will turn, out to be nine, we enter zero. Let's call it Cantor's number.

The idea of this proof is that you can construct a new number that is not in this sequence, in fact, we only construct a number that differs only from the first numbers in the sequence and not from all of them. This is a very important observation that undermines this proof. By constructing such a number, we are sure that there is no same number among those that are at the beginning of the sequence, but what about infinity quantity the numbers that are at the further places? Let's assume that we want to build a Cantor's number to the n number in the sequence that differs by the n decimal digit. The combination of all n digits is 10^n , which is a finite number. It may be that all numbers containing combinations of the first n digits are already present in our sequence at further places. For example, for the third digit:

$$1 \leftrightarrow 0{,}294732\ \ldots$$
$$2 \leftrightarrow 0{,}378820\ \ldots$$
$$3 \leftrightarrow 0{,}515682\ \ldots$$
$$\ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots$$

What will be the Cantor’s number ? 0,38? … if

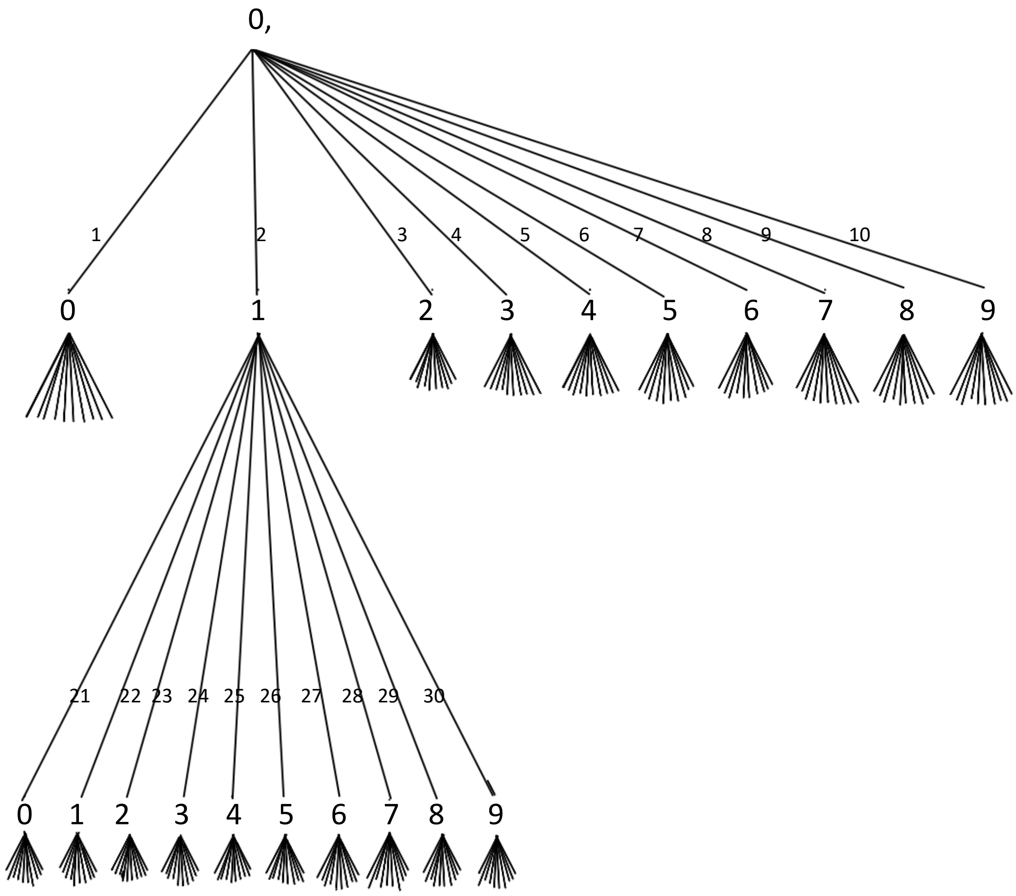
$$\begin{gathered} \dots\dots\dots \\ 11 \leftrightarrow 0{,}381\dots \\ 12 \leftrightarrow 0{,}382\dots \\ 13 \leftrightarrow 0{,}383\dots \\ 14 \leftrightarrow 0{,}384\dots \\ 15 \leftrightarrow 0{,}385\dots \\ 16 \leftrightarrow 0{,}386\dots \\ 17 \leftrightarrow 0{,}387\dots \\ 18 \leftrightarrow 0{,}388\dots \\ 19 \leftrightarrow 0{,}389\dots \\ 20 \leftrightarrow 0{,}380\dots \\ \dots\dots\dots \end{gathered}$$

In that case, we would have to go to n+1 digit, but here the situation repeats itself so on. It turns out that when the assumption is true, it is impossible to construct a new number. The way of constructing the Cantor’s number only gives us the illusion that we can construct a number that is not in the sequence. This is because we only see the first numbers in the sequence, but we cannot see all of them because there are infinitely many of them. The method of constructing the Cantor’s number gives us the possibility of constructing a new number only for a finite sequence or one in which there are not all numbers from the interval [0 , 1) . If the assumption is true, then when we take any number from this sequence and change any digit, it will turn out that this new number already appears in our sequence because according to our assumption, all combinations of digits have been exhausted.

Even if we had the sequence without all numbers and we could create such a Cantor’s number, then we can change the order based on Hilbert’s hotel [2] to such that this number will have an assigned number from the set $\mathbb{N}$, i.e. we create a new sequence of numbers. Then we’d have to start creating a new Cantor’s number. Then we create a new order of numbers and so on. Therefore, it cannot be considered proof $|\mathbb{N}| < |\mathbb{R}|$. We can create more Cantor’s numbers and add them to the series until it becomes impossible. We could even start with one, any number in that numerical interval, and use Cantor’s method to construct successive numbers by including them in a series. Such constructing of consecutive numbers its countability. After infinitely many steps, we would get a sequence of all the numbers in this range. In this way, we create a series of all real numbers from this set, which shows that this is feasible. This is one way we can rank this set. A bit in spiteful of Cantor. Diagonalization arguments [3] are often also the source of contradictions like Russell’s [4] paradox and Richard’s paradox.

REAL NUMBERS

Let’s create a tree that represents the numbers of the numerical interval $[0\,,1)$. Each number in this range has the form $0, a_1 a_2 a_3$ … As we can see, the quantity of digits is countable. There may be different ten digits in each place after the comma. We start with a zero to which we assign ten digits, and then to each of them another ten digits, and so on.



Each of these branches of the tree is a certain real number. Two things are important to us here. One is that the tree contains all the numbers $\mathbb{R}$ in the numerical interval $[0\,,1)$. The second is that each level has a finite number of branches, so each of them can be assigned a natural number. Thus, we obtained the order of all the numbers in this range. Numbers that are branches of this tree are numbered multiple times. To avoid this, we can perform a procedure. Note that a given branch whose natural number is assigned is clearly defined upwards but downwards it can be any part of the branch. Therefore, we can choose a given branch, e.g. a number $\pi - 3$, and going down do not assign its next natural number only numbering the next branch. Similarly, we proceed by leaving repetitive numbers in the matrix of rational numbers, proving their computability. By doing so we obtain the sequence of all numbers from the numerical interval $[0\,,1)$. We can write it like this on the first level:

$$
\begin{array}{cc}
1 \leftrightarrow 0{,}0\,\ldots & 0,-0-\cdots \\
2 \leftrightarrow 0{,}1\,\ldots & 0,-1-\cdots \\
3 \leftrightarrow 0{,}2\,\ldots & 0,-2-\cdots \\
\ldots & \ldots \\
10 \leftrightarrow 0{,}9\,\ldots & 0,-9-\cdots
\end{array}
$$

Going to the next level we add further digits to those numbers that we have and add further numbers:

$$
\begin{array}{ll}
1 \leftrightarrow 0{,}00\ldots & 0,-0-0-\cdots \\
2 \leftrightarrow 0{,}10\ldots & 0,-1-0-\cdots \\
\ldots & \ldots \\
10 \leftrightarrow 0{,}90\ldots & 0,-9-0-\cdots \\
11 \leftrightarrow 0{,}00\ldots & 0,-0-0-\cdots \\
12 \leftrightarrow 0{,}01\ldots & 0,-0-1-\cdots \\
13 \leftrightarrow 0{,}02\ldots & 0,-0-2-\cdots \\
\ldots & \ldots \\
20 \leftrightarrow 0{,}09\ldots & 0,-0-9-\cdots \\
21 \leftrightarrow 0{,}10\ldots & 0,-1-0-\cdots \\
22 \leftrightarrow 0{,}11\ldots & 0,-1-1-\cdots \\
23 \leftrightarrow 0{,}12\ldots & 0,-1-2-\cdots \\
\ldots & \ldots \\
109 \leftrightarrow 0{,}98\ldots & 0,-9-8-\cdots \\
110 \leftrightarrow 0{,}99\ldots & 0,-9-9-\cdots \\
\end{array}
$$

...

It seems that there are more natural numbers than real numbers because e.g. 1 and 11, 2 and 21 repeat themselves, but we can skip some of them. Initially, the quantity of numbers grows faster than the quantity of digits, but the number of digits in a given number is countability, which means in infinite there is the same quantity as elements of the sequence. After infinitely many steps we get all the numbers in the sequence and all the digits of these numbers. It is not possible to create a number that is not in this sequence because all combinations of digits have been exhausted. The set $[0\,,1)$ is equal to the set $\mathbb{R}$ which was proven by Cantor. This proves that the set of real numbers is a countability set.

CANTOR'S THEOREM for power set [5]

Each set has less power than the family of its subsets, i.e. power set.

Proof:
Let $f\colon A \to P(A)$ be any function from a given set A into its power set $P(A)$. Let's define a set B of those elements of set A that do not belong to their images in the function f :

$$B = \{x \in A\colon x \notin f(x)\}$$

Set B , as a subset of set A, is an element of power set A:

$$B \subseteq A \implies B \in P(A)$$

Therefore, for any element m belonging to set A , there is:

$$m \notin f(m) \implies m \notin f(m) \wedge m \in B \implies f(m) \neq B \qquad (1.1)$$

$$m \in f(m) \implies m \in f(m) \wedge m \notin B \implies f(m) \neq B \qquad (1.2)$$

Thus, the set B is not an image of any element of the set A in the mapping f, hence the function f cannot be a surjection (the "onto" function), and in particular cannot be a bijection. This means that the sets A and $P(A)$ are not equal $|A| \neq |P(A)|$.

The proof of the theorem seems convincing, but only apparently.

It seems that set B denies the existence of the bijection of the set into the power set. In fact, the opposite is true, it is bijection that prevents the formation of set B. It is the function f that creates the set B and does not vice versa. When the function f is a bijection, then for every subset of set A there is some element for which this subset is the image. Also, for the subset B there is some element m. According to the definition of the set B it is not the image of its elements, i.e. $m \notin B$. However, this means that m satisfies the condition $x \notin f(x)$, therefore:
If f is bijection then

$$\exists m \in A \quad f(m) = B \Longrightarrow m \notin B \wedge m \in B$$

that means a contradiction. The set B does not exist, the bijection f does not define such a set. This is because f, by assigning an element from the set A to the subset B, breaks the condition of the subset axiom which says that the predicate defining the subset B cannot contain B. Therefore, this set cannot be created within the Zermelo -Fraenkel axiom [6]. Illustratively, when the function f assigns an element m to a subset of B_n containing only elements satisfying the predicate $m \notin f(m)$, it simultaneously "breaks" this subset by adding another element to it. Therefore, set B cannot be created. This is a version of Russell's paradox.

Can such a bijection exist? We can show that yes.
Assume that the function f is not a bijection and (1.1) and (1.2) hold. There are subsets B_n containing some (but not all) elements from the set A satisfying the predicate $m \notin f(m)$. Is there the largest set of B in which for every n $B_n \subset B$? For this function, yes. For function f there is no image of set B. Let's assume that all subsets that are elements $f(A)$ are images of some element from set A except B. We can create a new function such that:

$$g(a_n) = \begin{cases} B, & n = 1 \\ f(a_{n-1}), & n > 1 \end{cases}$$

Such a function is a bijection of sets A and $P(A)$. For the function g there are different sets of B_n but for each, there is an element from the set A for which B_n is an image. That is, there is no largest set of B_n for the function g. The predicate $x \notin g(x)$ determines a class of sets [7], not a set.

$$\forall i \exists j \ \ B_i \subset B_j$$

The series of sets of B_n determined by bijections is infinite and divergent, and there is no sum.

For finite sets it is impossible to construct the function g, i.e. the bijection sets A and $P(A)$ can only exist for sets having infinitely many elements. For finite sets, Cantor's theorem is true. Set B may exist for some functions, such as non-bijection and only such.

Let's consider the power set of natural numbers. The set $P(\mathbb{N})$ has two types of sets, finite sets and infinite sets. Let's designate them as $P`(\mathbb{N})$ and $P``(\mathbb{N})$ respectively. Finite sets are a countable quantity.

$$|P`(\mathbb{N})| = |\mathbb{N}^1| + |\mathbb{N}^2| + |\mathbb{N}^3| + \cdots = \aleph_0 + \aleph_0 + \aleph_0 + \cdots = \aleph_0$$

We can notice that:

$$\forall P`` \in P``(\mathbb{N}) \ \ \exists a_{n_1}, a_{n_2}, a_{n_3}, \ldots \in \mathbb{N} \ \ P`` = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\} \Longrightarrow$$
$$\Longrightarrow \forall k \ \forall a_{n_k} \in P`` \ \ \exists P`_{n_k} \in P`(\mathbb{N}) \quad P`_{n_k} = \{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$$

That is, for any finite number of elements of an infinite subset, there is a finite subset containing these elements and only these. This means that infinite sets cannot be more than finite sets. Intuitive is like there are as many singletons as there are elements in a given set. Otherwise, there would have to be the largest finite set. Then the infinite subset would have to consist of several finite subsets, and thus there would be more combinations so that it would be more infinite subsets than finite subsets. There is no such set because there is no greatest natural number. We can rank subset $P`(\mathbb{N})$:

$$P`(\mathbb{N}) = \{P`_1, P`_2, P`_3, \ldots\}$$

Let's define a function $h: \mathbb{N} \to P``(\mathbb{N})$

$$h(n) = \{P`` \in P``(\mathbb{N}), \quad P`_n \subset P`` \wedge \ P`` \neq h(1), h(2), \ldots, h(n-1)$$

It is a function that assigns to every natural number a certain infinite subset of the power set $P(\mathbb{N})$. For every natural number n there is a finite subset $P`_n$ of a predefined series and there is an infinite subset $P``$ in which $P`_n$ is contained. The second condition guarantees that it is a differential function. Since each infinite subset differs from the other infinite subsets by at least one element, there is a corresponding finite subset which differs, from the other finite subset by at least one element. This means that it's a function "onto" which assigns a natural number to every infinite subset. So, it's a bijection. It proves that in a power set, there is a countable number of infinite subsets. So, there is a function f:

$$f(n) = \begin{cases} P`_n \in P`(\mathbb{N}), & 2n-1 \\ P``_n \in P``(\mathbb{N}), & 2n \end{cases}$$

We can see an apparent paradox here. For finite sets, there are only finite subsets, and there are more of them than elements of the set, which is expressed by the formula $|P(A_n)| = 2^n$. On the other hand, for infinite sets, in addition to finite subsets, we have infinite subsets and there are as many of them as the elements of the set. This is because there is no greater quantity than infinitely many. Just as it may seem that there are more $\mathbb{Z}$ numbers than $\mathbb{N}$, but at infinity there are the same number of them.

We can therefore make a more general theorem.

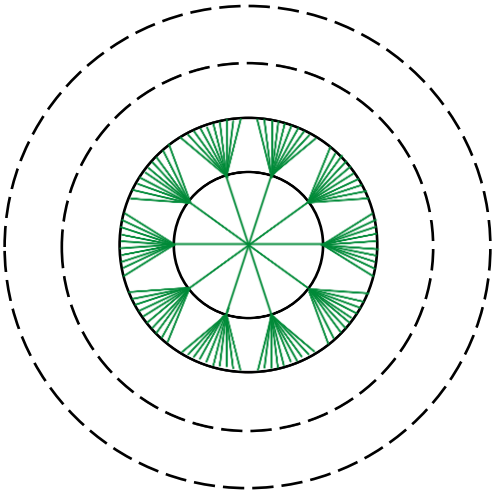
THEOREM

Any two sets of infinitely many elements are of the same power. Each set is countability.

$$\forall A_\infty \forall B_\infty \ \ |A_\infty| = |B_\infty| \qquad\qquad \forall A \ \ |A| \leq \aleph_0$$

This solves the so-called continuum hypothesis problem. This hypothesis assumes that Cantor's theorem about the existence of smaller and larger infinities is true. Since it is not so then naturally the question of which infinity is the next after the infinity of natural numbers loses its meaning. There is no cardinal number continuum $\mathfrak{c}$ or anything greater than aleph zero. Quantitatively, there is nothing more than infinity. Set means certain elements that we treat as a whole, but it does not determine where these elements are located or how they relate to each other. This means that we can "take" any element of the set and put it in the first place in the sequence. Then we can put any other element of the set in this series and so on. We don't need any function for this, we can select elements completely chaotically. There are no mathematical rules to prevent us from doing so and there is no element that we cannot place in a sequence, so any set is countability. The countability of a set follows directly from the axiom of pairing and the axiom of infinity. We can therefore say that the countability of a set is something very fundamental and results from the very nature of sets.

We can show equinumerous of sets $\mathbb{N}$ and $\mathbb{R}$ in another way. Let's imagine a countable number of concentric circles, each one larger than the last one. We mark the centre of the circle as 0, . On the first, the smallest, we mark ten points and connect the centre of the circles with these points by marking them with digits from 0 to 9. Each of these points is connected by successive with ten different points on the next circle. We do the same with the next circles, until infinity. In subsequent circles, the number of marked points and connecting episodes increases by an order of magnitude. Thus, a kind of web is formed, and each of the paths leading from the centre of the circles determines some real number from the numerical interval $[0\,,1)$. The number of points and paths is countable and can be numbered with natural numbers. In infinity, we will get all the numbers $\mathbb{R}$ from this numerical interval, and also all the points on the circle will be marked. This is because, as previously proven, there are the same number of points on each circle, regardless of its size. If one of the points on a given circle is not marked, it will be on the next one. If the number of points on a circle increased with its radius, we would not be able to determine whether all of them are marked with a digit. Unmarked points on a circle could only exist if the number of circles was finite. Therefore, countability is a fundamental property of infinity. This shows that the points on the circle are countability, and those in turn can be combined with points on the line that can be paired with real numbers.



In many mathematical equations, we can find such a fraction $\frac{1}{\infty}$. It is assumed that this fraction tends to zero. We write it like this

$$\frac{1}{\infty} \to 0$$

This means that this fraction is infinitely close to the number zero. This shows that there is a relationship between infinity and the number zero. If there were smaller and larger infinities as Cantor claimed, then it would have to be:

$$\aleph_0 < \aleph_1 \quad \Rightarrow \quad \frac{1}{\aleph_0} > \frac{1}{\aleph_1}$$

$$\downarrow \qquad \downarrow$$

$$\text{that is} \qquad 0_0 > 0_1$$

That means there would have to be smaller and larger numbers zero. The number zero is identified with the empty set. Within the axioms ZF there is only one empty set. There are no two different empty sets, so there are no smaller and larger empty sets. Similarly, there are no smaller and larger zero numbers. It follows that our assumption $\aleph_0 < \aleph_1$ is false. If, however, both fractions tended to the same number zero, then by would mean that they differ from each other by an infinitesimal amount, therefore these cardinal numbers at infinity would have to be equal.

$$\frac{1}{\infty} \rightarrow 0 \Rightarrow \text{only one: } \emptyset , 0 , \infty$$

Reference:

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