A THEOREM IN THE TRAPEZOID

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Abstract : In this note, we prove a remarkable theorem in the trapezoid.

<u>Theorem</u>

Let $ABCD$ be a trapezoid such that $AB = a$, $AD = c$, $DC = b$, $CB = d$, $AC = p$ and $BD = q$, we have :

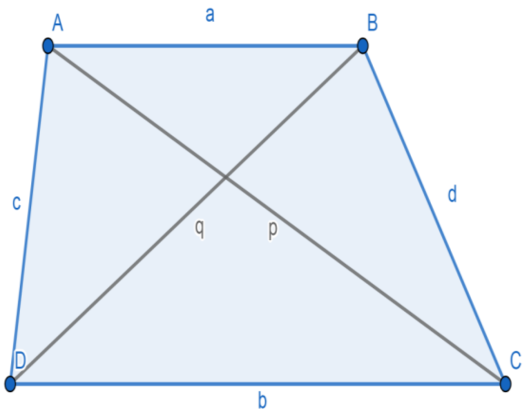
$$\begin{cases} a = \sqrt{\dfrac{(d^2 - p^2)^2 - (c^2 - q^2)^2}{2(d^2 + p^2) - 2(c^2 + q^2)}} & (1) \\ b = \sqrt{\dfrac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2(c^2 + p^2) - 2(d^2 + q^2)}} & (2) \end{cases}$$

This theorem help us finding any trapezoid's bases by knowing only his diagonals and legs.

<u>Proof</u>

We are going to prove formula (2).

We construct the trapezoid $ABCD$.



Heron's formula for the area of a triangle with lengths a, b, c is given by :

$$\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b++c)(a+b-c)} \qquad (3)$$

One can easily prove that the expression (3) is equivalent to the following expression :

$$\frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \qquad (4)$$

Now, using formula (4), the area K_1 of the triangle ACD in the above trapezoid is:

$$K_1=\frac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-p^2)^2}$$

Similarly the area K_2 of the triangle BCD is:

$$K_2=\frac{1}{4}\sqrt{4b^2d^2-(b^2+d^2-q^2)^2}$$

Since the two triangles ACD and BCD have the same base and altitude, their areas are equal, therefore :

$$K_1=K_2 \iff \frac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-p^2)^2}=\frac{1}{4}\sqrt{4b^2d^2-(b^2+d^2-q^2)^2}$$

$$\iff 4b^2c^2-(b^2+c^2-p^2)^2=4b^2d^2-(b^2+d^2-q^2)^2$$

$$\iff 4b^2c^2-4b^2d^2=-(b^2+d^2-q^2)^2+(b^2+c^2-p^2)^2$$

$$\iff 4b^2c^2-4b^2d^2=(b^2+c^2-p^2+b^2+d^2-q^2)(b^2+c^2-p^2-b^2-d^2+q^2)$$

$$\iff 4b^2c^2-4b^2d^2=(2b^2+c^2-p^2+d^2-q^2)(c^2-p^2-d^2+q^2)$$

$$\iff 4b^2c^2-4b^2d^2=2b^2(c^2-p^2-d^2+q^2)+(c^2-p^2+d^2-q^2)(c^2-p^2-d^2+q^2)$$

$$\iff 4b^2c^2-4b^2d^2-2b^2(c^2-p^2-d^2+q^2)=(c^2-p^2+d^2-q^2)(c^2-p^2-d^2+q^2)$$

$$\iff b^2(4c^2-4d^2-2c^2+2p^2+2d^2-2q^2)=(c^2-p^2+d^2-q^2)(c^2-p^2-d^2+q^2)$$

$$\iff b^2(2c^2+2p^2-2d^2-2q^2)=(c^2-p^2)^2-(d^2-q^2)^2$$

$$\iff b^2=\frac{(c^2-p^2)^2-(d^2-q^2)^2}{2c^2+2p^2-2d^2-2q^2}$$

$$\iff b=\sqrt{\frac{(c^2-p^2)^2-(d^2-q^2)^2}{2(c^2+p^2)-2(d^2+q^2)}}$$

Similarly we prove formula (1) by considering the triangles ABD and ABC.

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