EXPANSION OF THE GRAVITATIONAL POTENTIAL OF THE PROLATE HOMOGENEOUS ELLIPSOID IN SPHERICAL MULTIPOLES

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Abstract. In 1956 Schmidt represented the gravitational potential of a spheroid filled with a mass of homogeneous density by integrating the Newtonion potential over the entire spheroid. The drawback of this representation is that it depends on an angular parameter which satisfies an implicit equation in cylindrical coordinates. Using Maple as a concurrent tool to re-expand the terms in spherical coordinates, we derive the multipole expansion of these potentials up to 26th order in the inverse distance to the center.

1. Ellipsoid with Homogeneous Mass Density

Heymann wrote down an integral of the gravitational potential Φ of the homogeneous ellipsoid [6]

$$\Phi = \pi\rho a^2 c \int_\lambda^\infty \left(1 - \frac{\bar\omega^2}{a^2+u} - \frac{z^2}{c^2+u}\right) \frac{du}{(a^2+u^2)\sqrt{c^2+u}} \tag{1}$$

where $\bar\omega$ is the distance to the axis and z the distance along the axis of the cylinder coordinate system. a and c are the major and minor semi-axes of the oblate ellipsoid. ρ is the mass density; $e = \sqrt{1-(c/a)^2}$ is the eccentricity. The volume of the ellipsoid is $\frac{4}{3}\pi a^2 c$, its total mass $M = \frac{4}{3}\rho\pi a^2 c$. He concluded 'womit die gestellte Aufgabe vollständig erledigt ist' (and the task at hand is completely done with). This did not stop Schmidt [10] from solving it in terms of an implicitly defined angle β

$$\bar\omega^2 \sin^2\beta + z^2 \tan^2\beta = a^2 e^2, \tag{2}$$

such that outside the ellipsoid

$$\Phi = 2\pi \frac{\sqrt{1-e^2}}{e} \rho a^2 \beta - \frac{1}{2}(\bar\omega K_{\bar\omega} + z K_z) \tag{3}$$

with two auxiliary functions

$$K_{\bar\omega} = 2\pi \frac{\sqrt{1-e^2}}{e^3} \rho \bar\omega (\beta - \sin\beta\cos\beta), \tag{4}$$

$$K_z = 4\pi \frac{\sqrt{1-e^2}}{e^3} \rho z (\tan\beta - \beta). \tag{5}$$

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(2) is a bi-quadratic equation for $\cos\beta$

$$\left(-1+\frac{z^2+a^2e^2}{\bar\omega^2}\right)\cos^2\beta+\cos^4\beta-(z/\bar\omega)^2=0. \tag{6}$$

$$\cos^2\beta=\frac{1}{2}-\frac{z^2+a^2e^2}{2\bar\omega^2}\pm\sqrt{\left(\frac{1}{2}-\frac{z^2+a^2e^2}{2\bar\omega^2}\right)^2+\left(\frac{z}{\bar\omega}\right)^2} \tag{7}$$
$$=\frac{1}{2}-\frac{z^2+a^2e^2}{2\bar\omega^2}\pm\frac{1}{2}\sqrt{\left(1-\frac{z^2+a^2e^2}{\bar\omega^2}\right)^2+\left(\frac{2z}{\bar\omega}\right)^2}.$$

Utilizing

$$\cos^2\beta=\frac{1}{2}+\frac{1}{2}\cos(2\beta) \tag{8}$$

to avoid nested radicals and dropping the sign with imaginary solutions, this is written as [7]

$$\cos(2\beta)=-\frac{z^2+(ae)^2}{\bar\omega^2}+\sqrt{\left(1+\frac{z^2+(ae)^2}{\bar\omega^2}\right)^2-\left(\frac{2ae}{\bar\omega}\right)^2}. \tag{9}$$

2. Transition to Spherical Coordinates

The aim is to express the potential in spherical coordinates with θ the polar angle (co-latitude) and r the distance to the center of the ellipsoid:

$$z=r\cos\theta;\quad \bar\omega=r\sin\theta. \tag{10}$$

Remark 1. *Providing multipoles in spheroidal coordinates is discussed by Milo [9, 5]. Rewriting in spherical coordinates has benefits: orbits for an overlay of masses with different eccentricities, for examples planets and their rings, can be calculated from an overlay of the multipoles.*

The Taylor expansion of this cosine in powers of $1/r$ is

$$\cos(2\beta)=-\frac{\cos^2\theta+(ae/r)^2}{\sin^2\theta}+\frac{1}{\sin^2\theta}\sqrt{[1+(ae/r)^2]^2-(2ae/r)^2\sin^2\theta} \tag{11}$$
$$=-\frac{\cos^2\theta+(ae/r)^2}{\sin^2\theta}+\frac{1}{\sin^2\theta}\sqrt{[1-(ae/r)^2]^2+(2ae/r)^2\cos^2\theta}$$
$$=1-2\frac{(ae)^2}{r^2}+2\cos^2\theta\frac{(ae)^4}{r^4}+2\cos^2\theta(1-2\cos^2\theta)\frac{(ea)^6}{r^6}+2\cos^2\theta(1-5\cos^2\theta+5\cos^4\theta)\frac{(ea)^8}{r^8}$$
$$+2\cos^2\theta\left(1-9\cos^2\theta+21\cos^4\theta-14\cos^6\theta\right)\frac{(ea)^{10}}{r^{10}}$$
$$+2\cos^2\theta\left(1-14\cos^2\theta+56\cos^4\theta-84\cos^6\theta+42\cos^8\theta\right)\frac{(ea)^{12}}{r^{12}}+\cdots$$

More integer coefficients of these $\cos\theta$ polynomials and their exact representation by products of binomials are in the Online Encyclopedia of Integer Sequences [4, A033282]. The transition from the square root to that infinite series has been elucidated by Beckwith [1, (4)].

Remark 2. *The expansion of the squared sine is very similar*

$$\sin^2\beta = \frac{1}{2} - \frac{1}{2}\cos(2\beta) \tag{12}$$
$$= \frac{(ae)^2}{r^2} - \cos^2\theta\frac{(ae)^4}{r^4} - \cos^2\theta(1-2\cos^2\theta)\frac{(ea)^6}{r^6} - \cos^2\theta(1-5\cos^2\theta+5\cos^4\theta)\frac{(ea)^8}{r^8}$$
$$- \cos^2\theta(1-9\cos^2\theta+21\cos^4\theta-14\cos^6\theta)\frac{(ea)^{10}}{r^{10}}$$
$$- \cos^2\theta(1-14\cos^2\theta+56\cos^4\theta-84\cos^6\theta+42\cos^8\theta)\frac{(ea)^{12}}{r^{12}} + \cdots$$

Numerical estimates of $\sin\beta$ *can be started from the initial* $\sin\beta \approx ae/r$ *and iterating with* (2) *rewritten as*

$$\sin\beta = \frac{ea/r}{\sqrt{1-\sin^2\theta\sin^2\beta + (ea/r)^2}}. \tag{13}$$

To establish a power series for β itself, this is cross-matched with

$$\cos(2\beta) = 1 - 2\beta^2 + \frac{2}{3}\beta^4 - \frac{4}{45}\beta^6 + \frac{2}{315}\beta^8 + \cdots \tag{14}$$

by reversion of the power series, coefficients available in [4, A002544]. The multipole expansion of β results:

$$\beta = P_0(\cos\theta)\frac{ea}{r} - \frac{1}{3}P_2(\cos\theta)(\frac{ea}{r})^3 + \frac{1}{5}P_4(\cos\theta)(\frac{ea}{r})^5 - \frac{1}{7}P_6(\cos\theta)(\frac{ea}{r})^7 + \frac{1}{9}P_8(\cos\theta)(\frac{ea}{r})^9 - \cdots \tag{15}$$

3. Expansion in $1/r$ series

Introducing the standard sign for attractive potentials and including the gravitational constant G, Schmidt's potential outside the ellipsoid is

$$U = -G\Phi = -\frac{3GM}{2ea}\beta + \frac{1}{2}\left[\frac{3GM}{2e^3a^3}\bar{\omega}^2(\beta - \sin\beta\cos\beta) + \frac{3GM}{e^3a^3}z^2(\tan\beta - \beta)\right] \tag{16}$$
$$= -\frac{3GM}{2ea}\beta + \frac{1}{2}\left[\frac{3GM}{2e^3a^3}r^2\sin\theta^2(\beta - \sin\beta\cos\beta) + \frac{3GM}{e^3a^3}r^2\cos^2\theta(\tan\beta - \beta)\right].$$

Intermediate multipole expansions of terms in this potential are

$$\beta - \sin\beta\cos\beta = \frac{2}{3}(ea/r)^3 + \left(\frac{1}{5} - \cos^2\theta\right)(ea/r)^5 + \left(\frac{3}{28} - \frac{3}{2}\cos^2\theta + \frac{9}{4}\cos^4\theta\right)(ea/r)^7 + \ldots \tag{17}$$

and

$$\frac{1}{2}\sin^2\theta(\beta - \sin\beta\cos\beta) + \cos^2\theta(\tan\beta - \beta) = \tag{18}$$
$$= \frac{1}{3}(ea/r)^3 + \frac{1}{10}\left(-3\cos^2\theta + 1\right)(ea/r)^5 + \frac{1}{56}\left(3 - 30\cos^2\theta + 35\cos^4\theta\right)(ea/r)^7$$
$$+ \frac{1}{144}\left(105\cos^2\theta - 231\cos^6\theta + 315\cos^4\theta + 5\right)(ea/r)^9 + \cdots$$

which are plugged into the potential

(19)
$$U = -\frac{3GM}{2r}\beta(ea/r)+\frac{1}{2r}\left[\frac{3GM}{2e^3a^3}r^3\sin\theta^2(\beta-\sin\beta\cos\beta)+\frac{3GM}{e^3a^3}r^3\cos^2\theta(\tan\beta-\beta)\right]$$
$$= -\frac{3GM}{2r}\left[\beta(ea/r)-\left[\frac{1}{2e^3a^3}r^3\sin\theta^2(\beta-\sin\beta\cos\beta)+\frac{1}{e^3a^3}r^3\cos^2\theta(\tan\beta-\beta)\right]\right]$$
$$= -\frac{3GM}{2r}\left[\frac{2}{3}+\frac{1}{15}(1-3\cos^2\theta)(ea/r)^2+\frac{1}{140}(3-30\cos^2\theta+35\cos^4\theta)(ea/r)^4+\cdots\right]$$
$$= -\frac{GM}{r}\left[1+\frac{1}{10}(1-3\cos^2\theta)(ea/r)^2+\frac{3}{280}(3-30\cos^2\theta+35\cos^4\theta)(ea/r)^4+\cdots\right]$$
$$= -\frac{GM}{r}\left[1-\frac{1}{5}P_2(\cos)(ea/r)^2+\frac{3}{35}P_4(\cos)(ea/r)^4-\frac{1}{21}P_6(\cos)(ea/r)^6+\cdots\right].$$

4. Results

The multipole expansion coefficients [8, (2.50)]

$$U = -\frac{GM}{r}\left[1+\sum_{n\ge 1}\sum_{k=0}^{n}\left(\frac{a}{r}\right)^n P_n^k(\cos\theta)\left(C_{n,k}\cos(k\varphi)+S_{n,k}\sin(k\varphi)\right)\right] \quad (20)$$

are usually tabulated by the negated coefficients $J_n \equiv -C_{n,0}$. So for our case of the homogeneous oblate ellipsoid these are

$$J_2=\frac{e^2}{5};\quad J_4=-\frac{3e^4}{35};\quad J_6=\frac{e^6}{21};\quad J_8=-\frac{e^8}{33};\quad J_{10}=\frac{3e^{10}}{143};$$
$$J_{12}=-\frac{e^{12}}{65};\quad J_{14}=\frac{e^{14}}{85};\quad J_{16}=-\frac{3e^{16}}{323};\quad J_{18}=\frac{e^{18}}{133};$$
$$J_{20}=-\frac{e^{20}}{161};\quad J_{22}=\frac{3e^{22}}{575};\quad J_{24}=-\frac{e^{24}}{225};\quad J_{26}=\frac{e^{26}}{261}. \quad (21)$$

J_2 to J_8 have been published before by Hofmeister et al. [7]. The point is that the sober derivation above includes the general direction at all times. [It never assumes that θ is either exactly $0°$ or exactly $90°$ but includes the full $\cos\theta$ dependence in all orders of ea/r. It never assumes that forces—gradients of the potential—point into any particular direction. It does not truncate the expansion (15).] The claim of Hofmeister et al. that the result is 'not compatible with a series of Legendre polynomials beyond $n = 1$' is not correct.

Remark 3. *Conway considered more general ellipsoids with non-homogeneous mass densities [2].*

Remark 4. *Publications that use the variable θ for the latitude write $P_n(\sin\theta)$ in the expansion. Vinti switches between these two options [11].*

Remark 5. *In Fitzpatrick's lecture notes and books, J_2 appears as $2\epsilon/5$, where $\epsilon = c/a$ is termed ellipticity [3]. For small ellipticity we have $e \approx \sqrt{2\epsilon}$, so that notation is compatible with the result here.*

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