Riemann sums of $\sin x$ and $\cos x$

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Abstract

Riemann integrals of the trigonometric functions $\sin x$ and $\cos x$ have been computed directly from the appropriate Riemann sums.

Introduction

Let us remind the two fundamental theorems of integral calculus. Let $F(x)$ be the area under the plot of $f(x)$ between the line $x = a$, x coordinate line and the Ox axis. By looking at the Figure 1. we can see that

$$\lim_{\Delta x \to 0} F(x + \Delta x) - F(x) = \lim_{\Delta x \to 0} f(x + \Delta x)\Delta x \tag{1}$$

Dividing the above equation by Δx we arrive at

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x) \tag{2}$$

$F'(x)$ is the derivative of $F(x)$. $F(x)$ is being called the primary function of $F'(x) = f(x)$.

The first theorem of integral calculus states that the area derivative $F'(x)$ is equal to the function $f(x)$ under the plot of which the area $F(x)$ is being computed.

From Figure 2. we see that

$$\begin{aligned} F(x_1) - F(x_0 = a) &= f(x_1)\Delta x_1 \\ F(x_2) - F(x_1) &= f(x_2)\Delta x_2 \\ &\dots \\ F(x_{N-1}) - F(x_{N-2}) &= f(x_{N-1})\Delta x_{N-1} \\ F(x_N = b) - F(x_{N-1}) &= f(x_N)\Delta x_N \end{aligned} \tag{3}$$

where $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \dots, N$.

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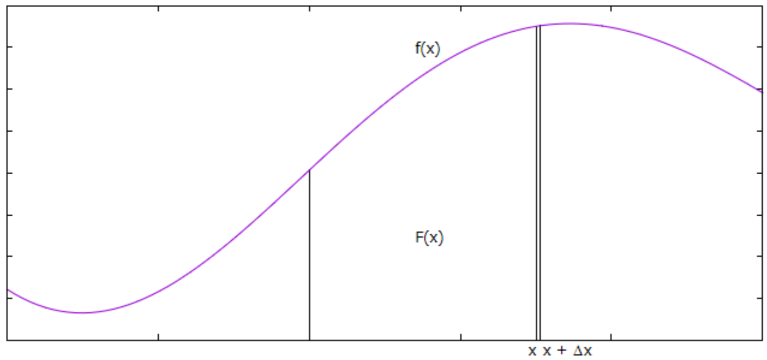

Figure 1: $F(x)$ is the measure of area under the plot of $f(x)$

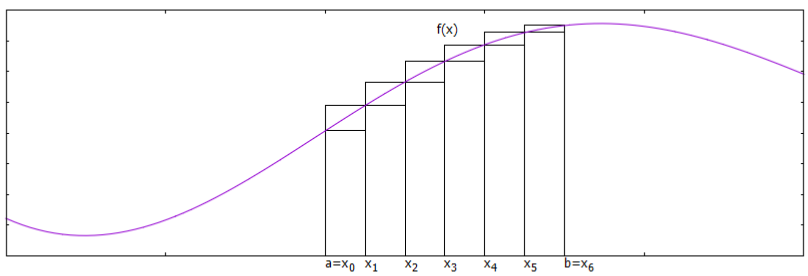

Figure 2: Illustration of the Riemann integral computation

Adding all above equations by sides we notice that certain terms cancel out and we arrive at the second theorem of integral calculus

$$F(b) - F(a) = \lim_{\Delta x_k \to 0} \sum_{k=1}^{N} f(x_k) \Delta x_k \tag{4}$$

The above equation on the right hand side has the (upper) Riemann sum for the function $f(x)$.

Computations

The Riemann sum of $\sin(x)$ and $\cos(x)$ we compute for

$$\Delta x_k = \Delta x = (b - a)/N \tag{5}$$

for $k = 1, 2, \ldots, N$ and choose it as

$$\lim_{\Delta x_k \to 0} \sum_{k=1}^{N} f(x_k)\Delta x_k = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(k\Delta x)\Delta x \tag{6}$$

with $x_k = k\Delta x$ for $k = 1, 2, \ldots, N$ so we have

$$F(b) - F(a) = F(N\Delta x) - F(0) \tag{7}$$

In order to compute the Riemann sums of $\sin x$ and $\cos x$ we use the formula for the sum of the geometric series as presented in [1]

$$\sum_{k=1}^{N} e^{ik\phi} = \sum_{k=1}^{N} \cos k\phi + i \sin k\phi \tag{8}$$

$$= \cos((N+1)\phi/2)\frac{\sin N\phi/2}{\sin \phi/2} + i \sin((N+1)\phi/2)\frac{\sin N\phi/2}{\sin \phi/2}$$

where $i^2 = -1$. We notice that

$$\cos\alpha \sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta)) \tag{9}$$

$$\sin\alpha \sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta)) \tag{10}$$

and we can substitute

$$\alpha = (N+1)\phi/2 \qquad \beta = N\phi/2 \tag{11}$$

$$\alpha + \beta = N\phi + \phi/2 \qquad \alpha - \beta = \phi/2 \tag{12}$$

After computations we arrive at

$$\sum_{k=1}^{N} \sin k\phi = -\frac{1}{2}(\cos(N\phi + \phi/2) - \cos\phi/2)\frac{1}{\sin\phi/2} \tag{13}$$

$$\sum_{k=1}^{N} \cos k\phi = \frac{1}{2}(\sin(N\phi + \phi/2) - \sin\phi/2)\frac{1}{\sin\phi/2} \tag{14}$$

Now we calculate the Riemann sums of $\sin x$ and $\cos x$

$$\sum_{k=1}^{N} \sin k\Delta x \cdot \Delta x = -\frac{\cos(N\Delta x + \Delta x/2)}{\sin \Delta x/2}\frac{\Delta x}{2} + \frac{\cos \Delta x/2}{\sin \Delta x/2}\frac{\Delta x}{2} \tag{15}$$

$$\sum_{k=1}^{N} \cos k\Delta x \cdot \Delta x = \frac{\sin(N\Delta x + \Delta x/2)}{\sin \Delta x/2}\frac{\Delta x}{2} - \frac{\sin \Delta x/2}{\sin \Delta x/2}\frac{\Delta x}{2} \tag{16}$$

We obtain

$$\lim_{\Delta x \to 0} \sum_{k=1}^{N} \sin k\Delta x \cdot \Delta x = \lim_{\Delta x \to 0} -\frac{\cos(N\Delta x + \Delta x/2)}{\frac{\sin \Delta x/2}{\Delta x/2}} + \frac{\cos \Delta x/2}{\frac{\sin \Delta x/2}{\Delta x/2}} \tag{17}$$

$$\lim_{\Delta x \to 0} \sum_{k=1}^{N} \cos k\Delta x \cdot \Delta x = \lim_{\Delta x \to 0} \frac{\sin(N\Delta x + \Delta x/2)}{\frac{\sin \Delta x/2}{\Delta x/2}} - \frac{\sin \Delta x/2}{\frac{\sin \Delta x/2}{\Delta x/2}} \tag{18}$$

Conclusion

The limits of the expressions

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x/2}{\Delta x/2} = 1 \tag{19}$$

and the Riemann sums for $\sin x$ and $\cos x$ are

$$\lim_{\Delta x \to 0} \sum_{k=1}^{N} \sin k\Delta x \cdot \Delta x = \lim_{\Delta x \to 0} [-\cos x]_{x=0}^{x=N\Delta x} \tag{20}$$

$$\lim_{\Delta x \to 0} \sum_{k=1}^{N} \cos k\Delta x \cdot \Delta x = \lim_{\Delta x \to 0} [\sin x]_{x=0}^{x=N\Delta x} \tag{21}$$

We can check that our results are correct. The derivative of $-\cos x$ is equal to the primary function $\sin x$ and the derivative of $\sin x$ is equal to the primary function $\cos x$.

References

[1] Hirst, Keith E. (1995) Modular Mathematics *Numbers, Sequences and Series* Butterworth-Heinemann, Oxford