The method of layered separation of n-cubes along the main diagonal and its application in geometry

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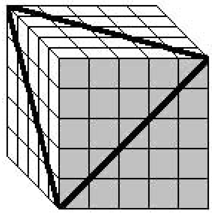
The non-obvious possibility of decomposing any n-cube consisting of n-cubes (including visually perceptible 2D and 3D) into layers of these cubes sequentially placed along the main diagonal of this n-cube is presented. At the same time, the number of *n*-cubes in each layer turned out to be closely related to the numbers of Pascal's triangle.

The coefficients of cutting each *n*-cube from the last *(n-1)* layers of them with a section of dimension *(n-1)D* are calculated.
Examples are given that allow us to outline some ways to further explore this possibility.
In Addition, the possibility of using this method to prove the tetrahedron volume formula without using infinitesimal methods is shown.

Let's start with a fairly simple example:



A secant plane is made in an ordinary 5x5x5 cube. Question: how many cubes will she cut?
If you can count them in a minute by removing one layer of cubes from the top (or bottom) of a large cube, then you have an excellent spatial imagination.

Fig.1

The second question is more complicated: a *4*-cube 5x5x5x5 is cut by a "flat" *3D* section (this will be a regular tetrahedron) by analogy with the previous one.
The question is the same: how many *4*-cubes will this section cut?

Visibility obviously cannot help here due to our lack of experience in observing in *4D*, and without any mathematical calculation method, it will not be possible to answer the question.

I managed to discover the possibility of layering an *n*-cube along its main diagonal into layers of its constituent *n*-cubes.

For a two-dimensional space:

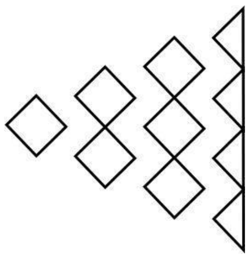


Fig.2 Half of the square is decomposed into layers of squares

$$\frac{4^2}{2} = 1 + 2 + 3 + \frac{1}{2}4$$

Integers show the number of squares in each layer, and the fraction 1/2 is the coefficient of cutting the squares of the last layer.

For *3D*:

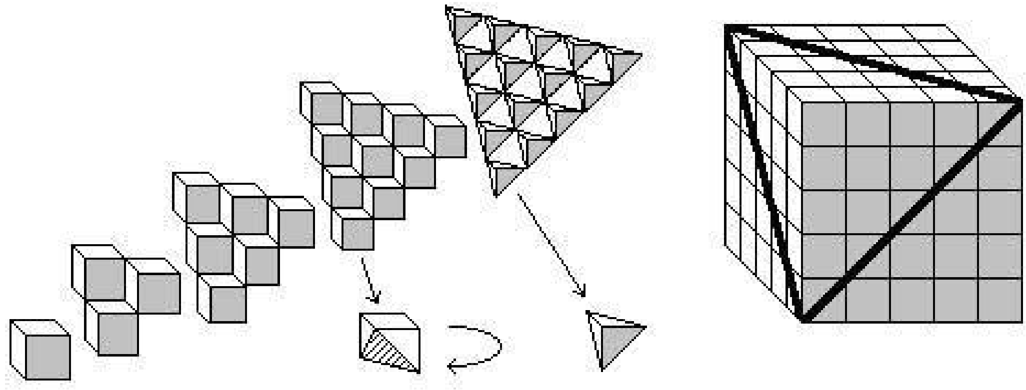


Fig.3 One sixth of the cube is decomposed into layers of cubes

$$\frac{5^3}{6} = 1 + 3 + 6 + \frac{5}{6}10 + \frac{1}{6}15$$

Integers show the number of cubes in each layer, which is closely related to the numbers of Pascal's triangle, and fractions 5/6 and 1/6 are the coefficients of cutting the cubes of the penultimate and last layers.
At the same time, we get the answer to our first question – 25 cubes will be cut.

In *4D* and above, visibility disappears, but the following approach can be used:
"When designing all the vertices of an *n*-cube onto one of its main diagonals, the latter will be divided by them into n identical parts" (*D.Hilbert, S. Cohn-Fossen* Visual Geometry).
In this case, the number of vertices of the *n*-cube projected at each point of the main diagonal represents the nth row of Pascal's triangle: for a square – 1 2 1, for a cube – 1 3 3 1 , for a 4-cube – 1 4 6 4 1 etc.

Any *n*-cube can be cut into x^n identical n-cubes (elements), where x is the number of cuts of each edge of a large cube.
This will allow us to bypass the need for a visual representation of the *n*-cube itself and its layers of elements. We will analyze only the relative position of the main diagonals of these elements in the form of successive layers along the main diagonal of the *n*-cube.

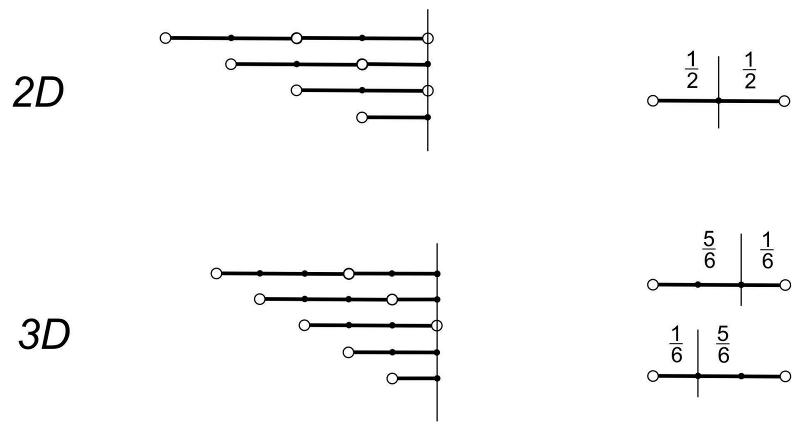


Fig. 4 The main diagonals of the elements for the above square and cube in Fig. 2 and 3.
We observe the displacement of the main diagonals of the elements, of which 1/2 square and 1/6 cube are made up, in each subsequent layer. The cutting coefficients of the elements of successive layers are shown on the right.

Let's show the location of the main diagonals of the elements of the *4*-cube.
Since the second layer of elements is tightly connected (vertex to vertex, edge to edge, face to face ...) with the original one, consisting of one *4*-cube, the main diagonals of the elements of the second layer will be shifted relative to the original one by 1/4 part. Each subsequent layer shifts its main diagonals relative to the previous one by the same amount. And only after four offsets, the diagonals of the current and original layers will be aligned.

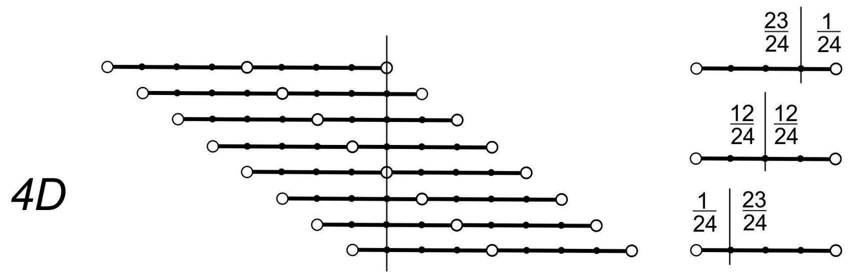


Fig.5 A *4*-dimensional cube, looking directly at the main diagonals of its elements. Only one diagonal of each layer is shown. The partition coefficients of the elements of successive layers are shown on the right

Replacing the number 4 with n in these arguments, you can see the arrangement of the layers of elements for any *n*-cube.
It follows that for any *nD*, the decomposition of an *n*-cube into layers of elements will be similar in terms of clarity to *2D* and *3D*, and a section of dimension *(n-1)D* will always cut *(n-1)* layers of elements.

Let's confirm our reasoning with the following formula for a *4*-cube of x^4 elements:

$$\frac{x^4}{24} = \frac{x(x-1)(x-2)(x-3)}{24} + \frac{23}{24}\frac{x(x-1)(x-2)}{6} + \frac{12}{24}\frac{(x+1)x(x-1)}{6} + \frac{1}{24}\frac{(x+2)(x+1)x}{6}$$

The first term is the number of uncut *4*–cubes, and fractions 23/24, 12/24 and 1/24 are the coefficients of cutting three subsequent layers of *4*-cubes along the main diagonal of a *4*-cube with an edge length x.
Opening the brackets and bringing similar ones, we are convinced of the fairness of equality.

It is not difficult to see the answer to the second question: for a *4*-cube at x= 5, 65 *4*-cubes will be cut (in layers 10+20+35).

For each *nD*, there is a set of coefficients for cutting *n*-cubes, which are very conveniently located on Pascal's triangle.

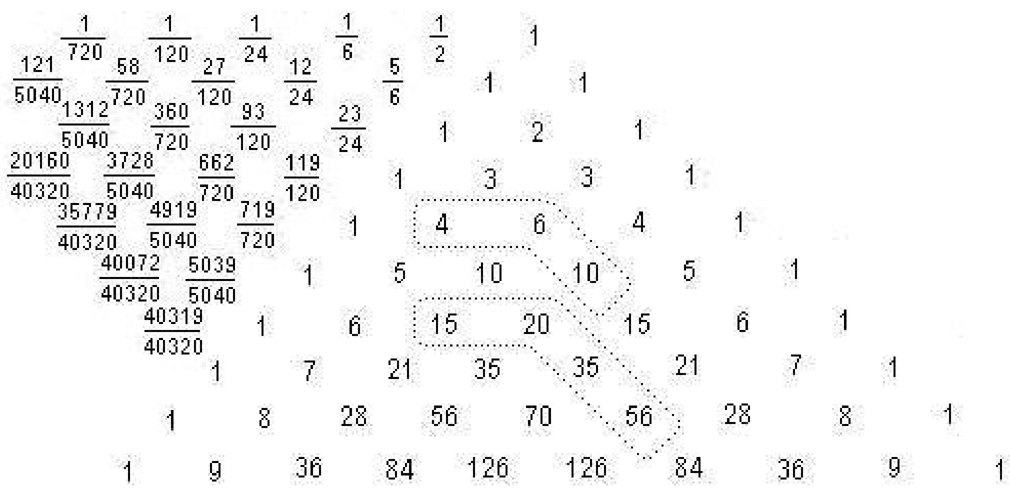


Fig.6 Pascal's triangle with *n*-cube cutting coefficients

Calculating the coefficients for any *nD* is not difficult and begins with cutting the first layer consisting of one element (*n*-cube), for which the coefficient is 1/n!
The second coefficient is calculated from the analysis of two layers of elements and so on.

$$\frac{4^3}{6}=1+3+\frac{5}{6}6+\frac{1}{6}10 \qquad\qquad \frac{6^4}{24}=1+4+10+\frac{23}{24}20+\frac{12}{24}35+\frac{1}{24}56$$

These numerical examples are shown in Fig. 6, where they are highlighted with dashed lines.

If Fig. 2 is ordinary and understandable, then Fig. 3 is quite psychologically difficult. I only managed to see him while playing with a child with simple wooden cubes.

Let's give an example for an eight-dimensional space:

$$\frac{5^8}{8!}=\frac{35779}{40320}1+\frac{20160}{40320}8+\frac{4541}{40320}36+\frac{248}{40320}120+\frac{1}{40320}330$$

Here, fractional numbers are the coefficients of cutting *8*-dimensional cubes for each layer, and integers are the number of elements in each layer.

The second example will be visually geometric for a four-dimensional space.
For a *4*-cube 3x3x3x3, you can show a *3D* section view that cuts three layers of *4*-cubes.

$$\frac{3^4}{24} = \frac{23}{24}1 + \frac{12}{24}4 + \frac{1}{24}10$$

The section in this case should consist of fifteen elements: eleven tetrahedra and four octahedra. One tetrahedron will be located in the center of the secant tetrahedron, four octahedra will be adjacent to all four of its faces, and the remaining 10 tetrahedra will be outside, forming all four faces of the large tetrahedron.

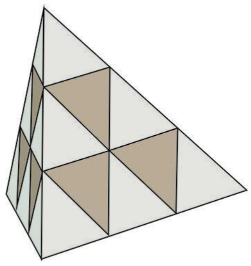


Fig.7 3D section view for a 4-cube 3x3x3x3

We consider that four tetrahedra are located in the four corners of a large tetrahedron and six more are in the middle of each of its edges. The four octahedra are highlighted in dark color and we observe only a part of their outer faces. The central small tetrahedron is not visible from any side of the large tetrahedron.

According to the following equality:

$$\frac{4^4}{24} = 1 + \frac{23}{24}4 + \frac{12}{24}10 + \frac{1}{24}20$$

let's show the view of the current tetrahedron for *4*-cube 4x4x4x4:

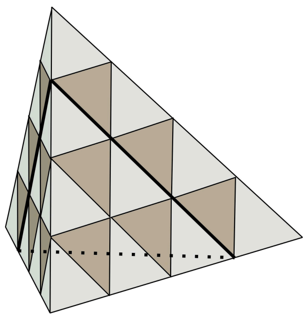


Fig.8 *3D* section view for *4*-cube 4x4x4x4

We note that it differs from the previous tetrahedron by only one additional layer of small 10 tetrahedra and 6 octahedra – it is highlighted by a plane with a bold perimeter. This can be done from any of its four sides. It has the appropriate symmetry.
You can also compare the flat appearance of any of the faces of this tetrahedron with the appearance of the face of an ordinary cube 4x4x4, obtained after removing four layers of cubes from it in the spirit of Fig.3. Note the complete coincidence of these species.
You can even try to imagine a *4D* section of a *5*-cube consisting of 4-tetrahedra and 4-octahedra. The analogy with Fig.8 helps.

The proposed **Method of layering an n-cube along the main diagonal** allows us to take a fresh look at the structural features of n-cubes and their representation in the form of successive layers of *n*-cubes.

Addition

I remember a drawing called "damn stairs" from school. I remember some of its strangeness and inconsistency between the complexity of the proof and the ultimate simplicity of the result.

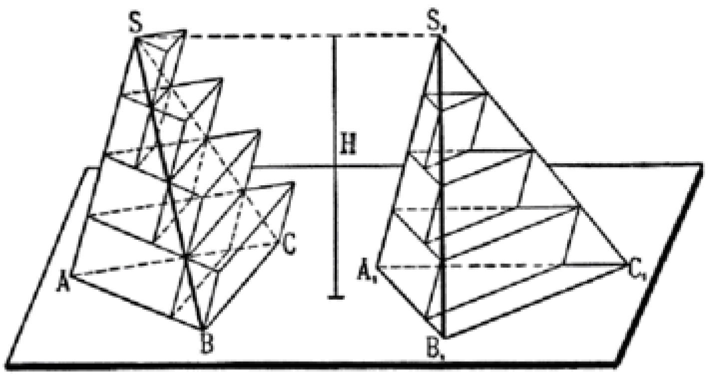


Fig.9

The method of exhaustion (infinite approximation), discovered in ancient times, proves the equality of the volumes of two pyramids with equal bases and the same heights. Ultimately, this lemma allows us to prove strictly that:

The volume of the pyramid is equal to one third of the product of the area of the base by the height (1).

Integral calculus can also be used to derive this formula.

In the formulation of his third problem, Hilbert, based on Gauss's letters to Christian Curling, raises the question: Is it possible to abandon the limit transition in the derivation of the formula for the volume of a triangular pyramid and limit oneself only to the method of equidistance. Max Den gave the first answer, and the answer is negative. Today, Hilbert's III problem is considered to be definitively closed.

What if mathematicians were just unlucky in finding a proof of formula (1) without going to the limit? (this is exactly how V. Boltyansky puts the question in his book "Hilbert's Third Problem").

This problem has always, since ancient times and up to the present day, been considered in the light of the relationship of only two objects – triangular prisms and pyramids having a common base and height.

Now we're going to take a bold step and complicate the task – add another prism as an object and see what happens. Indeed, sometimes the complication of the initial conditions can lead to a simplification of the decision process (recall D. Poya).

Let's add such a triangular prism so that, touching the first one, it would form a parallelepiped. We note that in this case it must be symmetrical to the existing prism, and therefore it will have an equal volume, which has been proven for a long time without involving integrals and "damn stairs".

Now we derive the formula for the volume of a triangular pyramid — tetrahedron, without the traditional transition to the limit.
The tetrahedron *ABCD* is the initial one, the volume of which must be determined.

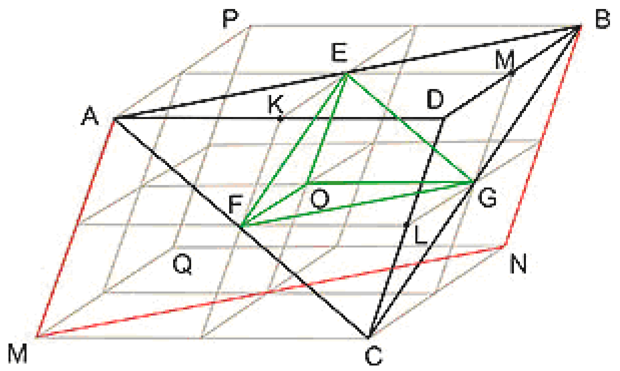


Fig.10

Let's add it to the parallelepiped as follows: we choose vertex *D* as the main one, and through vertices *A*, *B* and *C* we draw planes parallel to the opposite faces of the tetrahedron. And we will continue these faces of the tetrahedron until they intersect with the new planes.
We obtain a parallelepiped *DQ* consisting of two symmetrical prisms: *ABDMNC* and *APBMQN*. The first prism entirely contains the *ABCD* tetrahedron, the second one, which we added, does not contain it. The area of their *ABNM* junction is highlighted in red along the perimeter.
Divide the edges of the parallelepiped *AD*, *CD*, *BD* in half by the points *K*, *L*, *M* respectively. Through them we will draw new planes parallel to the faces of the parallelepiped. We get a parallelepiped *DQ* divided into eight small parallelepipeds, and all of them are similar to a parallelepiped *DQ*.

Let 's denote the volume of a small parallelepiped V_m . Then the volume of a large parallelepiped:

$$V_{DQ} = 8V_m$$

The initial tetrahedron consists of the following volumes: three small tetrahedra and a small parallelepiped without the same tetrahedron, but symmetrical to it (highlighted in green in Fig.10).

$$V_{ABCD} = V_{AEFK} + V_{EBGM} + V_{FGCL} + (V_{DO} - V_{EGFO})$$

Assuming (we will prove below) that the ratio of the volume of the tetrahedron and the parallelepiped (hereinafter referred to as *t*) does not change during the similarity transformation, we can write the following:

$$8V_m t = 3V_m t + (V_m - V_m t)$$

Reducing everything by V_m , we get:

$$t = \frac{1}{6}$$

That is, the tetrahedron occupies 1/6 of the parallelepiped, which in turn consists of two triangular prisms of the same volume.
Therefore, any tetrahedron occupies 1/3 of the volume of a triangular prism having a common base and height with it.

Previously, we used the statement that the ratio of the volume of a tetrahedron to the volume of the parallelepiped in which it is highlighted does not depend on the similarity transformation.

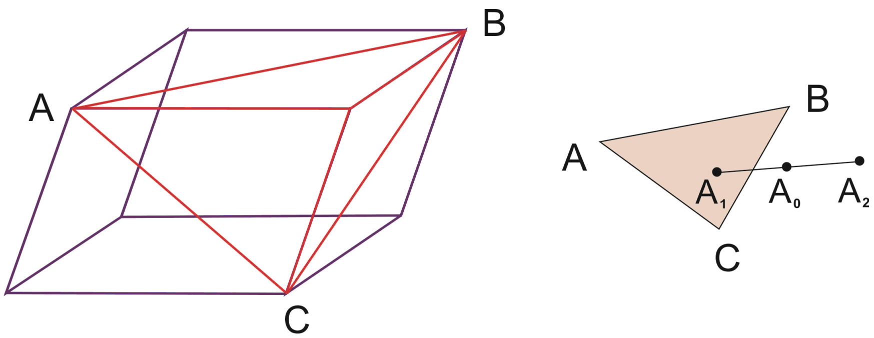


Fig.11

Suppose that this is not the case, and some part of the inner points of the tetrahedron, during the similarity transformation, passes into the points of the remaining part of the parallelepiped, i.e. the ratio of their volumes is violated.

Let's choose any of these transition points of the tetrahedron, let it be a point A_0 .
Let's lower the perpendicular from it to the plane *ABC* separating the volumes under consideration, we get a point A_1 . Let's continue the perpendicular beyond the point A_0 away from the plane *ABC*, without going beyond the tetrahedron. We will get a point A_2 .

So, the similarity transformation asserts, as one of the consequences, that the order of points on any straight line cannot change during the similarity transformation.

Thus, the point A_0 will never be able to cross to the other side of the *ABC* plane.
Similarly, it is proved that none of the points on the other side of the plane can pass into a tetrahedron.
Consequently, the ratio of the volumes of the considered polyhedra does not change during the similarity transformation, and we have the right to use this fact.

By the way, the requirement that the segment A_0A_1 is perpendicular to the *ABC* plane is unnecessary, since the basic definition of similarity allows you to choose any pair of points, the distance between which changes with a coefficient k during the similarity transformation.

If we choose the first point A_0 among certain ones (presumably passing through the *ABC* plane), then the second one A_1 can be any of the points in the *ABC* plane.

This proof is based on the following remarkable fact: only a cube and a parallelepiped can be composed of smaller, similar polyhedra. The triangular prism and other polyhedra do not have this ability.

Thus, the **Method of dividing the n-cube along its main diagonal** allowed us to approach the ancient proof here in an unexpected way.

Literature

D.Hilbert, S.Konfossen Visual Geometry, Moscow, Nauka, 1981.
V.Boltyansky Hilbert's Third Problem, Moscow, Nauka, 1977.