Applicability Operator Approach to Mathematical Proofs

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Abstract

Teaching proofs of theorems and encouraging students to both comprehend written proofs and originate their own can at times be a difficult undertaking. This is due in part to the lack of a single unifying process by which one can approach mathematical proofs. In this paper a method using set theory as a foundation is presented.

1 Introduction

Mathematical theorems can be stated as rules describing properties and relations between classes of mathematical objects such as numbers ,sets, matricies, topologies and others from various branches. For example, the theorem that integers greater than 2 can be factored as finitely many primes is a rule describing the relation between integers and their factorizations. It seems natural that we should turn to set theory to provide a formalization of this process , since it provides a logical method of describing various interrelations between sets of objects. To this end we need to define a few things:

2 Definitions

Set builder notation is simply the mathematical notation used to define sets. It is written as such: A={ x | statement about x }, as an example we have A=$\{x \in R | x^2 > 4\}$, ($\in$ read as element of, R read as the Reals) which reads set A is equal to the set of all x, an element of the Reals such that the square of x is greater than four.

Applicability Space: This is the set of all entities to which the theorem applies. For example in the prime factorization theorem mentioned above, the applicability space contains all integers greater than two. In the case of the theorem that says all nXn matricies with nonzero determinant are invertible, the applicability space would contain all nXn matricies. The theorem then acts as an operator on the applicability space, which leads us to the next definition.

Condition Operator: An operator can be described as a process which maps one thing to another. A condition operator is a mapping described by a set of rules. An example of a condition operator would be "the set of all blue cars driven by 25 year olds." When applied to the set of all cars, it isolates those cars satisfying the condition. A theorem can be viewed as a condition operator because it isolates the set of all mathematical entities for which the theorem is satisfied.

Theorem Space: This is the set of all mathematical entities for which the theorem is satisfied.

With this in mind we can state that the action of proving a theorem is equivalent to establishing an equality between Applicability Space and Theorem Space. If every element of Applicability Space is also mirrored in Theorem Space, than this is equivalent to saying that the theorem is true.

3 Examples applying this methodology

Theorem: *Every integer greater than 2 can be factored as finitely many primes.*

In this case the applicability space **A** is the set of all integers greater than two, the theorem space is the set of all integers greater than two which can be factored as finitely many primes, and the condition operator is all those members of the applicability space **A** which can be factored as finitely many primes. In all cases the theorem space will be a subset of applicability space. The condition operator is the selection of all members of **A** which can be factored as finitely many primes.

In this particular case, every member of the applicability space is also a member of the theorem space. Therefore we can say that the theorem is true.

Theorem: *Every nXn matrix with nonzero determinant is invertible.*

Like the prime factorization example, the applicability space can be read right from the statement of the theorem. It is the space of all nXn matricies with nonzero determinants. The theorem space is the subset of the applicability space for which the matricies are invertible. To prove this theorem requires we find equality between the two spaces. The condition operator is the process of selecting all members of the applicability space which are invertible.

Theorem: *For a given power series* $\sum_{i=1}^{n} a_n(x-c)$ there are only 3 possibilities.

(a) The series converges only when x=c

(b) The series converges for all x

(c) There is a positive number R such that the series converges when |x-c|<R and diverges when |x-c|>R

Here the applicability space **A** is not as clear as in the two previous examples, nonetheless it can be read from the statement of the theorem. It is the set of all power series in the above form. The theorem space is the space for which all the bottom three statements are true, and the condition operator is the selection of all the members of **A** for which the statements are true. Again, proving this theorem will involve finding an equality between the two spaces.
Diagrammed here:

A=Applicability Space—Set of all power series of the form $\sum_{i=1}^{n} a_n(x-c)$

CO=Condition Operator—The selection of members of A for which a, b, and c are true.

T=Theorem Space—The set of all power series for which all a, b, and c are true.

While veiwing theorems in this manner does not give a general method of proof, it does provide an approach which may yield insights not ordinarily realized. An insight may come by proving a similar but easier to prove theorem and showing that the theorem operator combination is isomorphic to the original one.

As an example of this, let us examine the simple trigonometric equality. The theorem is:

$Cos^2(\alpha) = \frac{1}{2} + \frac{Cos(2\alpha)}{2}$

The applicability space is $\mathbf{A} = \{\alpha \in \mathbb{R} | 0 < \alpha < 2\pi$, the selection operator is the above equality, and the theorem space is the space for which the equality is true. Now it can be shown that the set of all angles α is isomorphic to the set of all complex numbers of absolute value equal to one. Knowing this, and knowing the connection between trigonometric and complex numbers we can pvove the above.

The connection is the standard euler formula

$e^{i\theta} = Cos(\alpha) + i * Sin(\beta)$ and Re(z)= $\frac{z+\bar{z}}{2}$

We know $Cos(\alpha) = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$ So $Cos^2(\alpha) = \frac{(e^{i\alpha}+e^{-i\alpha})^2}{2}$

$= \frac{e^{i\alpha^2}+e^{-i\alpha^2}+2e^{i\alpha}e^{-i\alpha}}{4}$

$= \frac{e^{2i\alpha}+e^{-2i\alpha}+2e^{0}}{4}$

$= \frac{e^{2i\alpha}+e^{-2i\alpha}}{4} + \frac{1}{2}$ from the connection condition this equals $\frac{Cos(2\alpha)}{2} + \frac{1}{2}$

The relation is therefore proven in the space of complex numbers and therefore proven by an isomorphism between exponentials and the standard trig functions. It is these features that make this approach both interesting and useful.