Hydrodynamics as U(1) gauge theory

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We propose that hydrodynamics could be treated as $U(1)$ gauge theory where the velocity field written using Clebsch variables and the related vorticity are identical to the gauge potential and the field strength tensor, respectively.

Keywords: hydrodynamics, $U(1)$ gauge theory, velocity field, vorticity, Clebsch variables.

Let us observe\textsuperscript{1}

$$C(\bar{v}) \equiv \int \varepsilon^{ijk} \bar{v}^i \partial_j \bar{v}^k \, dr = \int \bar{v} \cdot \bar{\omega} \, dr$$

where $C(\bar{v})$ is the fluid helicity, $\bar{v}$ is velocity field, $\bar{\omega}$ is vorticity.

In 3-dimensional space, the Chern-Simons integral could be written as\textsuperscript{2–4}

$$h = \int_M \varepsilon^{\alpha\mu\nu} \bar{A}_\alpha \bar{F}_{\mu\nu} \, d^3r$$

where $h$ is the electromagnetic helicity, a non-zero integer number (if $h$ is zero it implies zero energy), $M$ denotes 3-dimensional manifold, $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$ denote the 3-dimensional space, $\bar{A}_\alpha$ is the gauge potential, $\bar{F}_{\mu\nu}$ is the gauge field tensor\textsuperscript{3} (the field strength tensor).

We consider that eqs. (1) and (2) are identical where velocity field and vorticity are identical to the gauge potential and the field strength tensor, respectively. It has a consequence that we could treat the hydrodynamics as the gauge theory.

In the simple case, let us assume that the hydrodynamical system is the fluid flow that has zero viscosity (an inviscid fluid) with non-zero vorticity in the Eulerian reference frame where the Clebsch representation for the velocity field is used. Then such a hydrodynamical system could be treated as an Abelian $U(1)$ gauge theory.

This is because the structure of the velocity field in such a hydrodynamical system does not involve non-commuting interactions (there are no interactions) between the different directional components of the velocity field.

The Clebsch representation for the velocity field can be written as\textsuperscript{5,6}

$$\bar{v}_\nu = f \, \partial_\nu q$$

and the related vorticity is

$$\bar{\omega}_{\mu\nu} = \partial_\mu \bar{v}_\nu - \partial_\nu \bar{v}_\mu$$

where

$$f = -1/[2\pi(1 + \rho_c^2)]$$

$\rho_c$ is a constant amplitude so $f$ is also a constant. We consider $\rho_c$ as an analogy with a constant amplitude of electromagnetic wave in a vacuum space\textsuperscript{7}, $q$ is the phase. $f$ and $q$ are Clebsch variables, both are scalars.

The identical gauge potential and field strength tensor are

$$\bar{A}_\nu = f \, \partial_\nu q$$

and

$$\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$$

respectively.

We would like to thank Richard Tao Roni Hutagalung and AI (Chat GPT) for fruitful discussions.

Special thanks to beloved Juwita Armilia and Aliya Syauqina Hadi for much love. To Ibunda and Ayahanda, may Allah bless them with the Jannatul Firdaus.

This research is fully supported by self-funding.

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