Key to Leibnizian Mathematics

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ABSTRACT

Euclidean (Abstract) Mathematics: Mathematics based on a discrete model for Space. It is built on the assumption that a piece of space, called a point, with zero extent exists and that all other spatial entities are synthesised by aggregation of points.

Leibnizian Mathematics: Mathematics based on a continuous model for Space. It is built on the assumption that all spatial entities have extent, and that space can be analysed by isolating and delimiting pieces of continuous Space.

Theis relevant basic assumptions of Mathematics that lead to some of the salient features of Abstract Mathematics are stated. These features are then deduced from these assumptions in a fundamental way. Infinitesimals and Infinitesimal Numbers are introduced and then used in an example of a Riemann sum to create a contradiction that motivates the introduction of Leibnizian Mathematics as a model for Mathematics that is supplemental to Abstract Mathematics. Leibnizian Mathematics is then introduced by stating its basic assumptions. Lastly a list of the meanings of some words that are common to both models, but describe properties that differ between the models, is given. The reader is then referred to the document LEIBNIZIAN MATHEMATICS, accessible on the link given on page 8.

1. PROLOGUE

Different Mathematics result when different assumptions are made. For example:

Let a straight line and a point not on that line be given.

- **Euclidean Geometry** results if it is assumed that one, and only one, line parallel to the given line can be drawn through the given point. In Euclidean geometry the sum of the interior angles of a triangle is exactly 180°.
- **Riemannian Geometry** results if it is assumed that no line parallel to the given line can be drawn through the given point. In Riemannian Geometry the sum of the interior angles of a triangle is more than 180°.
- **Hyperbolic Geometry** (Lobachevsky Geometry) results if it is assumed that more than one line parallel to the given line can be drawn through the given point. In Hyperbolic Geometry the sum of the interior angles of a triangle is less than 180°.

2. UNIVERSAL ASSUMPTIONS

Universal assumptions about Space, held as true for all Mathematics:

- **U1**: Solids exist and extend in three dimensions.
- **U2**: A surface is the interface between two abutting solids and extends in two dimensions.
- **U3**: A line is the interface between two intersecting surfaces and extends in one dimension.
- **U4**: A point is the interface between two intersecting lines. It is a place in space and extends in no direction.

3. EUCLIDEAN ASSUMPTIONS

Assumptions about Space, particular to Abstract Mathematics (Here called Euclidean Mathematics)

- **E1**: Axiom of Euclid: A point exists and is a piece of space with zero extent.
- **E2**: A solid is a clump of points.
- **E3**: A surface is a single layer of points.
- **E4**: A line is a string of points.
- **E5**: The Real line: There is an order-preserving one to one mapping of the real numbers onto the points of a line.

4. ESSENTIAL ANATOMY OF EUCLIDEAN MATHEMATICS

Let Z be an index set and let

$$\{A_\alpha | \alpha \epsilon Z\}$$

be a set of points onto which Z is mapped one to one. Let

$$D = \sum_{\propto \epsilon Z} d(A_\alpha)$$

Where $d(A_\alpha)$ is the maximum diameter of the point A_α and therefore $d(A_\alpha) = 0$.

EA1) If Z is a finite set, then D is a finite sum of zeros and therefore D=0.

EA2) If Z is a countable set, then D is the limit as n tends to infinity of the partial sums to n terms; and these are all zero. Hence D=0.

EA3) But in this model D must be larger than zero when the points are to form a line of non-zero length. Therefore, to form a line of non-zero length, the cardinality of the set Z must necessarily be **more than countable** in this model**.** This necessitates the introduction of the concept 'more than countable' into Euclidean Mathematics. There must therefore also exist more than countable many points and thus there must exist more than countable many real numbers to form the real line in this model.

EA4) As in EA2) above, D would always be zero whenever the sum is obtained by taking the limit of finite or countable sums of zeros. Therefore, to get D to be larger than zero without taking a limit requires in this model that more than countable many actions (additions) must be performed one by one until the sum is complete. This introduces the essence of the axiom of choice into Euclidean Mathematics[i]. It also validates the assumption that in this model an irrational number is an infinite string of digits[ii], with "infinite" as defined in EA5, because it is possible to determine all the required digits down to the last. All this is obviously not possible in perceived reality, hence the name "Abstract Mathematics".

EA5) The words 'infinite' and 'infinity' both mean 'an integer larger than all Natural Numbers' in this model.

EA6) Euclidean space is complete: Every set of nested intervals of which the lengths converge to zero has a point as limit.

5. EXTENSION OF NOMENCLATURE

Let $\{A_0; A_1; A_2; A_3; \ldots\}$ be given points on a line and let $\{s_n = L(A_n,A_0): n=1, 2, 3, \ldots\}$ form a Cauchy sequence converging to zero where $s_n = L(A_n,A_0)$ is the length of the line between the points A_n and A_0 for all n.

Definition

The set of nested intervals $\{(A_n,A_0): n=1,2,3, \ldots\}$ as described above is called an **infinitesimal focussed on A_0**, and the Cauchy sequence $\{s_n; n=1, 2, 3, \ldots\}$, which belongs to the equivalence class of Cauchy sequences converging to zero, is called an **infinitesimal number**.

The extension of these definitions to the case where the point A_0 is internal to the intervals forming the set of nested intervals is trivial. This definition can be extended to areas and solids.

6. DICHOTOMY

The first way of forming a line of non-zero length from points was by stringing together more than countable many points of zero length and then adding their lengths - as was done in section 4 above for Euclidean Mathematics.

But an alternative way of forming a line of non-zero length from points would be (as is done in Calculus for the Riemann integral) to begin with a line of non-zero length and then divide it into ever shorter pieces. After doing this an infinite number of times (as can implicitly be done in Euclidean Mathematics according to EA4 above) the limits should all be single points and the line would have been transformed into a string of points.

Are these two ways equivalent?

7. DECIDER

An example can prove nothing, a counterexample can disprove anything.

Consider the Riemann Integral:

$$1 = \int_0^1 1 \cdot dx$$

Riemann sums can be formed by starting with an interval (line) of length one on the X-axis as partition zero, then form successive partitions by dividing each interval of the previous partition into three equal parts. In this way the n^{th} partition will consist of 3^n intervals, each of length 3^{-n}. If $x = a_i^n$ is at the centre of the i^{th} interval of the n^{th} partition, then

$$a_i^n = \frac{2i-1}{2} 3^{-n} \quad \text{For } i = 1, 2, 3, \ldots, 3^n \text{ and } n = 0, 1, 2, \ldots. \qquad [6A]$$

But the length of the whole interval is the sum of the lengths of the parts so that:

$$1 = \sum_{i=1}^{3^n} L(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad \text{for n=0, 1, 2, } \ldots.$$

Where:

$$L\left(a_i^n - \frac{1}{2}3^{-n} \, , \, a_i^n + \frac{1}{2}3^{-n} \right) = 3^{-n}$$

Is the length of the line (with a_i^n at its centre) between the points $a_i^n - \frac{1}{2}3^{-n}$ and $a_i^n + \frac{1}{2}3^{-n}$ on the real line.

Since this sum is the same for all values of n

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} L(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad [6B]$$

Thus, the right-hand side of 6B is in some vague way a kind of multiple (that goes to infinity) of interval lengths (that all go to zero), and it is therefore some kind of indefinite form $\infty \cdot 0$

But these partitions have two specific properties that can be shown rigorously to be true, but can easily be seen by drawing three lines of unit length below each other and marking the partitions on them:

Firstly, when a point is in the middle of one part of a partition, it will be in the middle of a part for all subsequent partitions. Secondly, a set of nested intervals can be formed by selecting from consecutive partitions intervals having the same midpoint. The lengths of these intervals converge to zero while they remain symmetric about their common midpoint. In Euclidean Mathematics the set of intervals will have this point as a limit.

A nested set of intervals of which the lengths converge to zero was defined above as an infinitesimal, and here every infinitesimal is focussed on the common midpoint of all the intervals forming the infinitesimal.

The set of all the infinitesimals formed as described above is a directed set where the pre-order is defined by: “A<B is true when the first interval (part) of the infinitesimal B is contained in an interval that is a part of A”. The infinitesimal that has the whole unit interval as first partition is the first element in this directed set. The

Riemann integral can then be defined as a net on the directed set of infinitesimals mapping onto a value in the real numbers (in this case the number one).

But this net also maps onto the set:

$$D = \{ a_i^n \text{: for some n and some i} \}$$

of points that are the limits of the infinitesimals in Euclidean Mathematics. Note that these points all represent rational numbers.

Thus, for the limit of the Riemann sums, an obvious notational change implies that for the set D:

$$1 = \sum_{\propto\epsilon D} d(A_\alpha) \qquad [A]$$

But D is a set of rational numbers and hence it is countable. Therefore, according to the property EA2 of Euclidean Mathematics

$$0 = \sum_{\propto\epsilon D} d(A_\alpha) \qquad [B]$$

The contradiction formed by the results [A] and [B] of this informal argument[iii] is resolved by:

Conclusion

A set of nested lines of which the lengths converge to zero may not have a point (or any spatial entity of zero length) as a limit.

But, for the set of nested intervals to have a point as limit is a fundamental property of Euclidean Mathematics because of EA6. Therefore, this result implies that the Riemann integral – and by implication all of Calculus – should not be part of Euclidean Mathematics (hence the name “Non-standard Analysis”).

Therefore:

Referring to the prologue, it is thus necessary to find a set of assumptions - different from those of Euclidean Mathematics – to form a model called Leibnizian Mathematics in which a set of nested intervals of which the lengths converge to zero does not have a limit of zero extent but is merely a never-ending sequence of nested intervals.

8. NOTE ON THE LIMITS OF LINES IN MATHEMATICS

A geometrical vector has both magnitude and direction and is represented as a straight line of which the length is the magnitude of the vector, and the direction is the direction of the vector. When the length of a vector converges to zero the limit is the zero vector which has zero magnitude and has an indeterminate direction.

When the length of a line converges to zero in Euclidean Mathematics, the limit is a point. The limit has no length, and the line's property of direction has completely disappeared. In Euclidean Mathematics the nature of a line therefore changes in the limit from being a one-dimensional line into being a zero-dimensional point (I.E. In vectorial terms it changes from being a vector to being a scalar). In Leibnizian Mathematics this discrepancy does not occur.

9. LEIBNIZIAN MATHEMATICS

Leibnizian Mathematics is based on an alternative set of assumptions about Space in which points are mere places in space indicated by the endpoints of lines. To form this model, the Euclidean Assumptions EA1 … EA6 of paragraph 3 above are supplanted by:

LEIBNIZIAN ASSUMPTIONS

- **L1**: **Axiom of Parmenides**: All spatial entities have non-zero extent.
- **L2**: Any solid, surface or line can always be divided.[iv]
- **L3**: When divided, the total extent of the parts equals the extent of the original.
- **L4**: **The Real line**: There is an order-preserving one to one mapping of the real numbers onto lines from the origin; mapping the magnitudes of the numbers onto the lengths of the lines.

In this model a point is merely the endpoint of a line as noted in **U4** above, and therefore the role of points in forming the model is taken over by lines. This model and its properties are developed in the document "LEIBNIZIAN MATHEMATICS" and is posted on viXra. It can be accessed using the link:

http://viXra.org/abs/2201.0175

The model is developed from page 9 onwards in this document (But please also take note of TO THE READER on page 8).

10. PRIMER

In Leibnizian Mathematics space is not synthesised from points, but is analysed by solids, surfaces and lines. It is never required to sum zeros to a non-zero total, and hence none of the conclusions of paragraph 4 are applicable to Leibnizian Mathematics. To ease access to the document "LEIBNIZIAN MATHEMATICS" referred to above, the following primer is a discussion of some consequential differences in the meaning of words common to both models as an effort to alleviate paradigm shock.

Mathematics

Euclidean (Abstract) Mathematics: Mathematics based on a discrete model for Space. It is built on the assumption that a piece of space, called a point, with zero extent exists and that all other spatial entities are formed through aggregation of points.

Leibnizian Mathematics: Mathematics based on a continuous model for Space. It is built on the assumption that all spatial entities have extent, and that space can be analysed by isolating and bounding pieces of continuous Space.

Zero

Euclidean: It can be defined as a number less than all positive numbers. As a real number it is also the equivalence class of Cauchy sequences converging to zero.

Leibnizian: Same as in Euclidean Mathematics. As a Cauchy (infinitesimal) number it is the null Cauchy sequence (0.; 0.0; 0.00; ….).

Infinity (Noun)

Euclidean: An integer larger than all other integers.

Leibnizian: An irrational number only; the equivalence class of divergent sequences. The individual divergent sequences in this class are the infinite Cauchy numbers.

Infinite (Adjective, Adverb)

Euclidean: Larger than all integers.

Leibnizian: Never ending, open ended, unbounded.

Infinite decimal fraction

Euclidean: A non-finite decimal fraction containing all its digits up to and including the last one.

Leibnizian: A proper name given to an equivalence class of Cauchy sequences without any other numerical connotation.

Point

Euclidean: A piece of space with zero extent.

Leibnizian: The endpoint of a line. (As such it cannot have intrinsic extent because it is not a spatial entity but a property of a line.)

Limit

Euclidean: A way of handling continuity in a model of Mathematics based on a discrete model of space.

Leibnizian: Absent - in the sense of being not required.

(See the adaptation of the rule of L'Hospital on page 18 of the reference.)

Cauchy Number

Euclidean: Absent.

Leibnizian: The Cauchy numbers are the sequences that form the equivalence classes that define the real numbers. They are classified as:

Infinitesimal numbers: The Cauchy sequences that form the real number zero.

Infinite Numbers: The divergent Sequences that form the real number infinity.

Rated numbers: The Cauchy sequences that form all other real numbers.

More than countable

Euclidean: Cardinality of the sets of points and real numbers

Leibnizian: Absent

11. CONCLUSION

Like with Geometry as mentioned in the prologue, Mathematics divides into two sub models namely Euclidean Mathematics that is more suitable for discrete problems (like in Algebra and in the Theory of Probability for discrete events that are modelled as non-spatial entities) and Leibnizian Mathematics that is more suitable for continuous problems (like in Calculus and in parts of Statistics).

Hence, they are different and independent Models, suitable to be used in different circumstances.

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[i] It is therefore mandatory, whenever using Euclidean Mathematics, to make sure that the Axiom of Choice is only applied in arguments solely about abstract entities.

[ii] In Cantor's well-known proof that there are more than countable many real numbers, his argument assumes that these numbers are countable and therefore that it is possible to make a list of all infinite decimal fractions. He then showed that there existed an infinite decimal fraction that was not in the list, and from that he concluded that **all** infinite decimal fractions cannot be listed and thus there must be more than countable many real numbers. But an equally valid conclusion is that the real numbers cannot be listed **at all** – namely that a list of a single infinite decimal fraction cannot be made (as in perceived reality). The conclusion EA4 ensures the existence of such a symbol and validates the proof, albeit valid only in abstract Euclidean space.

[iii] Even though the argument is informal and D dense in the unit interval, one should note that any number that is not of the form [6A] will eventually fall outside the intervals forming any given infinitesimal, and hence cannot be a limit for any infinitesimal. This gives the argument a claim to being formal even though it lacks a proper notation.

[iv] The well-known rhyme about fleas can be adapted to Leibnizian Mathematics:

Big space has little space
That sum to what is in it,
And little space has lesser space,
And so on without limit.