On the Geometrization of Quantum Mechanics, Frozen Stars, the Bohm-Poisson and Nonlinear Klein-Gordon Equations

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May 2024

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Abstract

We revisit the nonlinear Klein-Gordon-like equation that was proposed by us which capture how quantum mechanical probability densities curve spacetime, and find an exact solution that may appear to be "trivial" but with important physical implications related to the physics of frozen stars and with Mach's principle. The nonlinear Klein-Gordon-like equation is essentially the static spherically symmetric relativistic analog of the Newton-Schrödinger equation. We finalize by studying the higher dimensional generalizations of the nonlinear Klein-Gordon-like equation and examine the relativistic Bohm-Poisson equation as yet another equation capturing the interplay between quantum mechanical probability densities and gravity.

1 Introduction

Rather than quantizing geometry (gravity), the geometrization of quantum mechanics, if possible, is an appealing process. For instance, the emergence of quantum mechanics from the fractal geometry of spacetime was advanced long ago by Nottale [1] in his formulation of the scale relativity theory. Another approach in the interplay between gravity and quantum mechanics has been based on the Newton-Schrödinger equation [2], [4]. It is obtained by coupling the Schrödinger equation to the Poisson equation. The potential is the gravitational potential determined by the Poisson equation associated to the matter density, and which in turn, is proportional to the probability density corresponding to the wave-function, $\rho \sim \Psi^*\Psi$. The Newton-Schrödinger integro-differential equation is

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \;=\; -\;\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) \;+\; V(\vec{r},t)\;\Psi(\vec{r},t) \;-\; \left(Gm^2 \int \frac{|\Psi(\vec{r}',t)|^2}{|\vec{r}-\vec{r}'|}\; d^3r' \right)\;\Psi(\vec{r},t) \tag{1}$$

In [5] we found exact solutions to the stationary spherically symmetric Newton-Schrödinger equation in terms of integrals involving *generalized* Gaussians. The energy eigenvalues were also obtained in terms of these integrals which agree with the numerical results in the literature.

The authors [3] have shown that the Schrödinger-Newton equation for spherically symmetric gravitational fields can be derived in a WKB-like expansion from the Einstein-Klein-Gordon, and Einstein-Dirac-Cartan system. The derivation amounts to assuming the validity of the semi-classical Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle$. where the left hand side is treated classically, but in the right hand side one is taking the expectation value of the stress energy operator in the state $|\Psi\rangle$.

There are two descriptions (interpretations) of the physical process underlying the geometrization of QM. Both are based on the postulate that the quantum probability density can *curve* the classical spacetime. One description is based in Feynman's path integral formulation of QM, where the ensemble of paths (fluid-like trajectories) associated with a point-particle of mass M_o provides the probability density ρ (fluid-like density) which curves the (classical) spacetime background in which the point particle moves.

A related example is in the study of Bohm's formulation of QM, when Santamato [6] found that Bohm's quantum potential $Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}$ was proportional to the spatial part of the Weyl scalar curvature (in a $flat$ spacetime)[1] when the spatial components of the Weyl's gauge field of dilatations are pure gauge $A_i \sim \frac{\partial ln(\rho)}{\partial x^i}$ and where ρ is the ensemble density associated with the particle's paths.

The results of [6] were extended to the relativistic case with A_μ proportional to $\frac{\partial ln(\phi^* \phi)}{\partial x^\mu}$, where $\phi(x^\mu)$ is a complex scalar field leading to the Klein-Gordon equation. Because A_μ is a total derivative, the Weyl field strength is zero, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$, which implies that the rate of the ticking of clocks (in flat spacetime) will be independent of their paths taken from point A to B. Consequently, atomic clocks arriving on the earth via different trajectories will tick at the same rate (same spectral lines). In this fashion one can avoid Einstein's criticism of Weyl's geometry. The extension of the results in [6] to the Dirac equation and nonlinear quantum mechanics can be found in [7].

The other description (interpretation) underlying the geometrization of QM, and described in this work, is based on the gravitational field produced by $smearing$ a point mass M_o at $r = 0$ throughout all of space (in an spherically symmetric fashion) . The gravitational field is similar to the one generated by a self-gravitating anisotropic fluid droplet of energy-mass density $\rho = M_o \varphi^*(r)\varphi(r)$, and such that the latter energy-mass density stems from the Quantum Mechanical probability amplitude $\varphi(r)$ associated with a spinless point-particle of mass M_o. The smearing process resembles a probability density "cloud" $\varphi^*(r)\varphi(r)$ permeating all of the 3-spatial region $\mathcal{D}_3 = \int_0^\infty 4\pi r^2 dr$ at a given time t.

Classically one may smear the point mass in any way we wish leading to arbitrary density configurations $\rho(r)$. However, Quantum Mechanically this is *not* the case because the radial mass configuration $M(r)$ (which determines the density $\rho(r)$) must obey a key

[1] When the metric is flat the only contribution to the Weyl curvature stems from the Weyl gauge field A_μ

third order (in the *spatial* derivatives) nonlinear differential equation (nonlinear extension of the Klein-Gordon equation) displayed in this work and which is the static spherically symmetric relativistic analog of the Newton-Schrödinger equation.

The above-mentioned nonlinear Klein-Gordon-like equation was proposed in [20] but no solutions were presented. In section **2** we shall find an exact solution for $M(r)$ that may appear to be "trivial" but with important physical implications related to the physics of a frozen star [9] and with Mach's principle [12].

It is important to mention that a nonlinear Klein-Gordon equation can result from coupling a scalar field to gravity in cosmology. Assuming a homogeneous and isotropic universe whose metric is given by the Friedmann-Robertson-Walker metric involving the scaling factor $a(t)$, and a scalar field solely dependent on time $\phi(t)$, and a potential $V(\phi)$, the coupled system of equations provided by the Einstein field equations and the Klein-Gordon like equation for $\phi(t)$ will furnish the solutions of the form $a = a(t)$ and $\phi = \phi(t)$. Eliminating t from these two solutions yields the functional relation $a = a(\phi)$. If one inserts this relation back into the Klein-Gordon like equation for ϕ one will automatically generate a *nonlinear* equation for ϕ because the D'Alambertian operator $\Box$ depends on the metric, and in turn, on the scaling factor $a = a(\phi)$. As a sign of consistency, naturally, the nonlinear Klein-Gordon like equation will be automatically $satisfied$ when one plugs in the solution $\phi = \phi(t)$ found earlier when one solved the coupled system of equations. In the most general case, having found solutions to the coupled system of field equations, $g_{\mu\nu}(x^\sigma)$ and $\phi(x^\sigma)$, it is much harder to infer the functional relationship between $g_{\mu\nu}$ and ϕ that would have generated a nonlinear Klein-Gordon like equation for ϕ.

In section **3** we study the higher dimensional generalizations of the nonlinear Klein-Gordon-like equations displayed in section **2** and examine the relativistic Bohm-Poisson equation as yet another equation capturing the interplay between probability densities and gravity.

2 A Solution to the Nonlinear Klein-Gordon Equation and Frozen Stars

Throughout this work we shall be employing the units $\hbar = c = 1$. In this section we shall show how the quantum probability density can *curve* the classical spacetime and find a solution to the nonlinear Klein-Gordon equation which is related to the physics of frozen stars [9]. For simplicity, we focus on spherically symmetric static gravitational backgrounds. Let us start with the Schwarzschild-like static spherically symmetric metric

$$(ds)^2 \ = \ - \ (1 - \frac{2GM(r)}{r})(dt)^2 \ + \ (1 - \frac{2GM(r)}{r})^{-1}(dr)^2 \ + \ r^2(d\Omega_2)^2 \qquad (2)$$

based on a mass function $M(r)$ representing the mass enclosed in a spherical region of radius r. The metric (2) is *not* a solution of the vacuum field equations but instead is a solution to the Einstein field equations with sources $G^\mu_\nu = 8\pi G T^\mu_\nu$. The stress energy tensor is given by

$$T^\mu_\nu \;\equiv\; diag\;(-\rho(r),\; p_r(r),\; p_\theta(r),\; p_\varphi(r)) \tag{3}$$

and whose components turn out to be

$$\rho(r) \;=\; -\; p_r(r) \;=\; \frac{1}{4\pi r^2}\,\frac{dM(r)}{dr},\;\; p_\theta(r) \;=\; p_\varphi(r) \;=\; -\;\frac{1}{8\pi r}\,\frac{d^2M(r)}{dr^2} \tag{4}$$

The conservation law $\nabla_\mu T^\mu_\nu = 0$, after laborious algebra gives

$$p_\theta = p_\phi = -\;\rho\;-\frac{r}{2}\,\frac{d\rho}{dr} \tag{5}$$

which is consistent with (4).

To proceed next we shall follow very closely the construction of relativistic wave-functions by [8] (**not** to be confused with second-quantized fields in Quantum Field Theory). If a *one*-particle wave function can be denoted by $\Psi(x^\mu)$, it is natural to introduce the spacetime scalar product

$$< \Psi|\Psi > \;=\; \int d^4x\;\Psi^*(x^\mu)\;\Psi(x^\mu) \tag{6}$$

and to normalize Ψ such that

$$1 \;=\; \int d^4x\;\Psi^*(x^\mu)\;\Psi(x^\mu) \tag{7}$$

The quantity

$$dP_{(4)} \;=\; \Psi(x^\mu)^*\;\Psi(x^\mu)\;d^4x \tag{8}$$

is naturally interpreted as probability that the particle will be found in the (infinitesimal) spacetime 4-volume d^4x.

If eq-(8) is the fundamental 4-probability, then

$$\Psi^*_{(3)}(x^\mu)\;\Psi_{(3)}(x^\mu) \;=\; \frac{\Psi^*(x^\mu)\;\Psi(x^\mu)}{N_t},\;\; N_t \;=\; \int d^3x\;\Psi^*(x^\mu)\;\Psi(x^\mu) \tag{9}$$

can be interpreted as the conditional 3-probability such that

$$dP_{(3)} \;=\; \Psi_{(3)}(x^\mu)^*\;\Psi_{(3)}(x^\mu)\;d^3x \tag{10}$$

is the probability that the particle will be found in the (infinitesimal) 3-volume d^3x, in the case one knows that the particle is detected at time t. Since $\Psi(x^\mu)$ is normalized to unity one can infer that N_t is also the marginal probability that the particle will be found at time t over the whole 3-dimensional region $\Sigma_t = \int d^3x$.

Having briefly introduced the relativistic wave function proposal by [8] let us focus now in the case where Ψ can be decomposed (factorized) as

$$\Psi(x^\mu) \;=\; \varphi(\vec{x})\;\xi(t) \tag{11}$$

so that the 3-probability density

$$\Psi^*_{(3)}(\vec{x})\ \Psi_{(3)}(\vec{x}) \ = \ \frac{\varphi^*(\vec{x})\ \varphi(\vec{x})}{\int d^3x\ \varphi^*(\vec{x})\ \varphi(\vec{x})} \tag{12}$$

is independent on t and is automatically normalized to unity

$$1 \ = \ \int d^3x\ \Psi^*_{(3)}(\vec{x})\ \Psi_{(3)}(\vec{x}) \tag{13}$$

In the spherically symmetric case $\Psi(x^\mu) = \varphi(r)\xi(t)$, the overall normalization condition must be unity

$$1 \ = \ \int d^4x\ \Psi^*(r,t)\ \Psi(r,t) \ = \ \int_0^\infty \varphi^*(r)\ \varphi(r)\ 4\pi r^2\ dr\ \int_0^\infty \xi^*(t)\ \xi(t)\ dt \tag{14}$$

reflecting the fact that the probability of detecting the particle anywhere in the whole of 3-space, and along its entire world-line history, has to be unity. Eq-(14) leads to the two conditions

$$N \ = \ \int_0^\infty \varphi^*(r)\ \varphi(r)\ 4\pi r^2\ dr, \quad \frac{1}{N} \ = \ \int_0^\infty \xi^*(t)\ \xi(t)\ dt \tag{15}$$

We have taken the temporal domain's range from $t = 0$ to $t = \infty$. One could have taken it instead to range from $t = -\infty$ (infinite past) to $t = \infty$ (infinite future) . But for now we concentrate in the $former$ case. Given the mass $M(r)$ enclosed in the spherical region $0 \leq r' \leq r$

$$M(\varphi(r)) \ = \ M(r) \ = \ M_o \int_0^r \varphi^*(r')\ \varphi(r')\ 4\pi r'^2\ dr' \tag{16}$$

the D' Alambertian is given by

$$\Box \equiv \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\ g^{\mu\nu}\ \partial_\nu), \quad \hbar = c = 1 \tag{17}$$

and the analog of the Klein-Gordon-like equation is

$$(\ \Box \ - \ {M_o}^2\ )\ \Psi(x^\mu) \ = \ 0 \tag{18}$$

where, once again, $\Psi(x^\mu)$ must *not* be confused with the second-quantized scalar field $\Phi(x^\mu)$. Given the metric (2), the KG-like equation becomes

$$\frac{1}{r^2}\partial_r\left(\ r^2\ (1 - \frac{2GM(r)}{r})\ \xi(t)\ \partial_r\varphi(r)\ \right) \ - \\ \frac{1}{r^2}\partial_t\left(\ r^2\ (1 - \frac{2GM(r)}{r})^{-1}\ \varphi(r)\ \partial_t\xi(t)\ \right) \ - \ M_o^2\ \varphi(r)\ \xi(t) \ = \ 0 \tag{19}$$

The differential equation (19) has the form

$$A(r)\ \xi(t) \ - \ B(r)\ \partial_t^2\xi(t) \ - M_o^2\ \varphi(r)\ \xi(t) \ = \ 0 \tag{20}$$

separating the radial dependence from the temporal one yields an expression of the form

$$\frac{A(r) \ - \ M_o^2 \ \varphi(r)}{B(r)} \ = \ \frac{\partial_t^2 \xi(t)}{\xi(t)} \ = \ \lambda \tag{21}$$

since the left hand side solely depends on r, and the right hand side solely depends on t, they have to be both equal to a constant λ. One then can solve for $\xi(t)$

$$\xi(t) \ = \ \xi_o \ e^{t\sqrt{\lambda}} \tag{22}$$

Given the above solution for $\xi(t) = \xi_o \ e^{t\sqrt{\lambda}}$, it leads to the integro-differential equation

$$\frac{1}{r^2}\partial_r \left(\ r^2 \ (1 - \frac{2GM(r)}{r}) \ \partial_r \varphi(r) \ \right) \ -$$

$$\frac{\lambda}{r^2} \ \left(\ r^2 \ (1 - \frac{2GM(r)}{r})^{-1} \ \varphi(r) \ \right) \ - \ M_o^2 \ \varphi(r) \ = \ 0 \tag{23}$$

where the mass function $M(\varphi(r)) = M(r)$ is defined in terms of $\varphi(r)$ by the integral (16).

When $\varphi(r)$ is real-valued $\varphi^*(r) = \varphi(r)$, the above integro-differential equation (19) can be converted into a *nonlinear* differential equation involving the mass function $M(r)$, after expressing $\varphi(r)$ in terms of $M(r)$ via eq-(16), as follows

$$\frac{dM(r)}{dr} = M'(r) \ = \ 4\pi M_o r^2 \ \varphi^*(r) \ \varphi(r) \ = \ 4\pi M_o r^2 \ \varphi(r)^2 \ \Rightarrow \ \varphi(r) \ = \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \tag{24}$$

In this fashion, after writing $\varphi(r)$ in terms of $M'(r)$, one ends up with a complicated *third* order nonlinear differential equation for the mass function $M(r)$

$$\frac{1}{r^2}\partial_r \left(\ r^2 \ (1 - \frac{2GM(r)}{r}) \ \partial_r \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \ \right) \ -$$

$$\frac{\lambda}{r^2} \ \left(\ r^2 \ (1 - \frac{2GM(r)}{r})^{-1} \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \ \right) \ - \ M_o^2 \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \ = \ 0 \tag{25}$$

Thus, eqs-(16,19) can be rewritten in terms of a single equation eq-(25) representing the static spherically symmetric relativistic analog of the Newton-Schrödinger equation.

Suffice to say, to find non-trivial exact analytical solutions of the third order nonlinear differential equation (25) for the mass function $M(r)$ is very difficult. Nevertheless there is an exact solution for $M(r)$ that may appear to be "trivial" but with important physical implications related to the physics of a frozen star [9] and with Mach's principle [12]. We will show that a solution for the mass function obeying eq-(25) is given by

$$M(r) \ = \ \frac{\kappa}{2G} \ r \ \Rightarrow \ \rho(r) \ = \ \frac{\kappa}{8\pi G r^2} \tag{26}$$

and the corresponding metric (2) becomes

$$(ds)^2 \ = \ - \ (1-\kappa) \ (dt)^2 \ + \ (1-\kappa)^{-1} \ (dr)^2 \ + \ r^2 (d\Omega_2)^2 \tag{27}$$

where κ is a parameter whose range of values is chosen to be $0 < \kappa < 1$. if $\kappa > 1$ there will be a signature flip. If $\kappa = 1$ it leads to a degenerate metric. If $\kappa < 0$ it leads to negative mass density. If $\kappa = 0$ it leads to a flat metric. Thus on physical grounds one must have $0 < \kappa < 1$.

Contrary to appearances the metric (27) is *not* flat. Given a $4D$ metric of the form

$$(ds)^2 \;=\; -\; f(r)\; (dt)^2 \;+\; f(r)^{-1}\; (dr)^2 \;+\; r^2 (d\Omega_2)^2 \tag{28}$$

the scalar curvature is

$$\mathcal{R} \;=\; -\; \frac{d^2 f(r)}{dr^2} \;-\; \frac{4}{r}\,\frac{df}{r} \;+\; \frac{2}{r^2}\,(1 - f(r)) \tag{29}$$

When the metric function is $f(r) = 1 - \frac{2GM}{r} \Rightarrow \mathcal{R} = 0$ as expected in the Schwarzschild case. When $f(r) = 1 - \frac{\Lambda}{3} r^2 \Rightarrow \mathcal{R} = 4\Lambda$ as expected in the de Sitter (Anti de Sitter case when $\Lambda > 0$ ($\Lambda < 0$), respectively.

One can verify that the metric (27) leads to a non vanishing $\mathcal{R} = \frac{2\kappa}{r^2}$. Another way of showing why the metric (27) is not flat can be seen by performing a coordinate change $t \to t' = \sqrt{1-\kappa}\, t, \;\; r \to r' = \frac{r}{\sqrt{1-\kappa}}$ and leading to

$$(ds)^2 \;=\; -\; (dt')^2 \;+\; (dr')^2 \;+\; (1-\kappa)\; r'^2 (d\Omega_2)^2 \tag{30}$$

one can see that the *areal* radius is no longer r' but $\sqrt{1-\kappa}\, r'$. Thus the area enclosed by r' is no longer given by the flat space result $\pi r'^2$ but instead by $(1-\kappa)\pi r'^2$.

It is a remarkable coincidence that the metric (27), when κ is less but very close to 1, such that $1 - \kappa = \epsilon^2$, where ϵ is an extremely small parameter, corresponds to the metric of what is called a *frozen star* [9]. The frozen star is a ultracompact object that, to an external observer, looks exactly like a Schwarzschild black hole, but with a different interior geometry and matter composition. The frozen star needs to be sourced by an extremely anisotropic fluid, for which the sum of the radial pressure and energy density is either vanishing or perturbatively small. Given $M(r) = \frac{\kappa}{2G} r$ one finds indeed that

$$\rho(r) \;=\; -\; p_r(r) \;=\; \frac{1}{4\pi r^2}\,\frac{dM(r)}{dr} \;=\; \frac{\kappa}{8\pi G r^2} \;\Rightarrow\; \rho + p_r = 0 \tag{31a}$$

$$p_\theta(r) \;=\; p_\varphi(r) \;=\; -\; \frac{1}{8\pi r}\,\frac{d^2 M(r)}{dr^2} \;=\; 0 \tag{31b}$$

which is consistent with the findings in [9]. As shown above, the scalar curvature $\mathcal{R} = \frac{2\kappa}{r^2}$, and ρ, p_r are singular at $r = 0$ so that a very small concentric sphere must be regularized to ensure that these densities remain finite. This process was described in [10].

Also, as detailed in [9], a matching process is required near the outermost layer of the star so that the frozen star metric and its corresponding stress tensor in eqs-(31) match smoothly to the exterior Schwarzschild geometry. A salient feature of the maximally negative radial pressure is that the frozen star model is able to evade the singularity theorems of Hawking and Penrose [14]. For a finite value of $1 - \kappa = \epsilon^2$, a trapped surface is never actually formed and having $\rho + p_r = 0$ means that geodesics do not converge.

Thus, the conditions under which the singularity theorems are valid are not satisfied by the frozen star geometry [9]. The frozen star model differs from the *gravastar* model of [11] which have a de Sitter like core in the interior and surface tension on the boundary.

After this detour discussing some of the properties of frozen stars, one finds that upon inserting the solution $M(r) = \frac{\kappa}{2G} r$ into the third order nonlinear differential equation the first term of eq-(25) vanishes due to $\partial_r (r^2 \partial_r (\frac{1}{r})) = 0$, and one ends up with the last two terms

$$\lambda \ (1-\kappa)^{-1} \ + \ M_o^2 \ = \ 0 \ \Rightarrow \ \lambda \ = \ - \ M_o^2 \ (1-\kappa) < 0 \tag{32}$$

so that the temporal dependence is

$$\xi(t) \ = \ \xi_o \ e^{t\sqrt{\lambda}} \ = \ \xi_o \ e^{\pm i M_o \sqrt{1-\kappa} t} \ = \ \xi_o \ e^{\pm i \omega t}, \ \ \omega = M_o \sqrt{1-\kappa} \tag{33}$$

with $0 < \kappa < 1$.

Given that $M(r) = \frac{\kappa}{2G} r$, the radial part $\varphi(r)$ ends up being

$$\varphi^*(r) \ = \ \varphi(r) \ = \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \ = \ (\frac{\kappa}{8\pi G M_o r^2})^{1/2} \tag{34}$$

and, finally, a full solution for the relativistic wave-function obeying the nonlinear KG-like equation is

$$\Psi(r,t) \ = \ \sqrt{\frac{\kappa}{8\pi G M_o}} \ \frac{\xi_o}{r} \ e^{\pm i M_o \sqrt{1-\kappa} t}, \ \ 0 < \kappa < 1 \tag{35}$$

Ψ vanishes as $r \rightarrow \infty$ as expected but it oscillates in time. However, the norm $\Psi^* \Psi$ does not depend on time and one has then found a stationary solution.

Normalization Conditions and Mach's Principle

Going back to the normalization conditions of eqs-(15) one finds

$$N \ = \ \int_0^\infty \ \varphi^*(r) \ \varphi(r) \ 4\pi r^2 \ dr \ = \ \frac{\kappa}{2GM_o} \ \int_o^\infty dr \ = \ \infty \tag{36}$$

so N diverges. To remedy this one could introduce an infrared cut-off determined by the Hubble radius R_H such that

$$N \ = \ \frac{\kappa R_H}{2GM_o} \ = \ \frac{\kappa}{2} \ \frac{R_H}{L_P} \ \frac{M_P}{M_o}, \ \ G = L_P^2 \tag{37}$$

and where L_P, M_P are the Planck length and mass, respectively. Introducing an infrared cut-off is similar to what occurs in ordinary QM when one has plane-wave solutions. The latter wave functions are not square-integrable, unless one places the particle in a box of finite size.

The mass of the observed universe M_U, consistent with the Black-Hole Cosmology scenario [13], is related to the Hubble radius as follows $R_H = 2GM_U$. Inserting this relation into (36) yields

$$N \ = \ \frac{\kappa R_H}{2GM_o} \ = \ \kappa \ \frac{M_U}{M_o} \ = \ \frac{\kappa}{2} \ \frac{R_H}{L_P} \ \frac{M_P}{M_o} \ \Rightarrow \ M_U \sim 10^{60} M_P \tag{38}$$

Introducing now the normalization coefficient $\frac{1}{\sqrt{N}}$ into the solution for $\varphi(r)$ furnishes the following normalized solution

$$\varphi_{normalized}(r) \ = \ \frac{\varphi(r)}{\sqrt{N}} \ = \ \frac{1}{\sqrt{N}} \sqrt{\frac{\kappa}{8\pi G M_o}} \ \frac{1}{r}, \quad 0 < \kappa < 1 \tag{39}$$

The expression for N depends on the ratio $\frac{M_U}{M_o}$ involving the mass M_o of the particle and the observed universe's mass M_U. As the mass of the particle *varies* : $M_o \rightarrow M'_o$ so does the value of $N = \kappa \frac{M_U}{M_o} \rightarrow N' = \kappa \frac{M_U}{M'_o}$, and such that

$$1 \ = \ \frac{1}{N} \int_0^{R_H} \ \varphi^*(r) \ \varphi(r) \ 4\pi r^2 \ dr \ = \ \frac{1}{N'} \int_0^{R_H} \ \varphi^*(r) \ \varphi(r) \ 4\pi r^2 \ dr \ = \ \ldots \tag{40}$$

for *all* values of $M_o, M'_o, M''_o, \ldots$. If one desires to be more rigorous with the notation it requires writing the spatial part of the normalized wave function as

$$\varphi_{normalized}(r) \ = \ \frac{\varphi_{(M_o)}(r)}{\sqrt{N_{(M_o)}}} \tag{41}$$

in order to denote the explicit M_o dependence of the solutions in eq-(39). In particular, the normalization coefficient $N_{(M_o)} = \kappa \frac{M_U}{M_o}$ actually depends on the *ratio* of the mass of the universe M_U and the mass of the particle M_o. In this sense one can venture to contemplate a Machian-like behavior operating in these solutions due to the presence of M_U resulting from the introduction of the infrared cut-off R_H in the integral. It is very important to emphasize that the *only* matter present in all of these results *is* the mass of the particle M_o. There is no other matter besides that, despite the the normalization constant $N_{(M_o)} = \kappa \frac{M_U}{M_o}$ depends on the numerical value of M_U. If M_o is set to be equal to the mass of the observed universe M_U, and consistent with the Black-Hole Cosmology scenario, one has

$$N_{(M_U)} \ = \ \frac{\kappa R_H}{2 G M_U} \ = \ \kappa \tag{42}$$

The introduction of an infrared cut-off given by the Hubble radius R_H requires introducing also a temporal cut-off given by the Hubble time T_H, and such that the other normalization condition in (15) leads to

$$\frac{1}{N_{(M_o)}} \ = \ \int_0^{T_H} \xi^*(t) \ \xi(t) \ dt \ = \ |\xi_o|^2 \ T_H \ \Rightarrow \ |\xi_o|^2 \ = \ \frac{1}{N_{(M_o)} T_H} \tag{43}$$

and the marginal probability of finding the particle anywhere in the universe at time t is constant and inversely proportional to the Hubble time. As T_H increases, this marginal probability gets smaller and smaller. A scaling $\xi(t) \rightarrow \sqrt{N_{(M_o)}} \xi(t)$ will render the integral (43) to unity. Note that the $\sqrt{N_{(M_o)}}$ factors *decouple* from the relativistic wave function $\Psi(r,t)$ given by eq-(35)[2] since it must obey the condition (14).

[2]Multiplying and dividing by $\sqrt{N_{(M_o)}}$ leaves $\Psi(r,t)$ fixed

3 Higher Dimensional Generalizations and the Bohm-Poisson Equation Revisited

The higher-dimensional extension of the Schwarzschild metric was found by Tangherlini [15] and can be obtained by simply replacing $(d\Omega)^2 \rightarrow (d\Omega_{D-2})^2$ (the $D-2$-dim solid angle) and $1 - \frac{2GM}{r} \rightarrow 1 - (\frac{r_h}{r})^{D-3}$ where r_h is the horizon radius expressed in terms of M and the gravitational coupling G_D in D dimensions whose units are $(length)^{D-2}$. The higher dimensional metric is given by

$$ds^2 \ = \ - \ f(r) \ (dt)^2 \ + \ \frac{(dr)^2}{f(r)} \ + \ r^2 \ (d\Omega_{D-2})^2, \ \ f(r) \ = \ 1 \ - \ \frac{16\pi G_D M}{(D-2)\Omega_{D-2} r^{D-3}} \tag{44}$$

where G_D is the D-dim Newton's constant, M the black hole mass. The solid angle of a $D-2$-dim hypersphere is $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$. The horizon radius is determined from the condition $f(r_h) = 0$ giving

$$r_h \ = \ \left(\frac{16\pi G_D M}{(D-2) \ \Omega_{D-2}} \right)^{\frac{1}{D-3}} \tag{45}$$

such that the metric (44) can be rewritten as

$$ds^2 \ = \ - \ [\ 1 - (\frac{r_h}{r})^{D-3} \ ] \ (dt)^2 \ + \ [\ 1 - (\frac{r_h}{r})^{D-3} \ ]^{-1} \ (dr)^2 \ + \ r^2 \ (d\Omega_{D-2})^2 \tag{46}$$

One can repeat the whole procedure of section **2** by replacing $M \rightarrow \mathcal{M}(r)$ for an r-dependent mass function so that the metric function of the *modified* Tangherlini metric is now given by

$$f(r) \ = \ 1 \ - \ \frac{16\pi G_D \mathcal{M}(r)}{(D-2)\Omega_{D-2} r^{D-3}} \tag{47}$$

The relation between the energy-mass density and the probability density $\varphi^*(r)\varphi(r)$ is now of the form

$$\rho(r) \ = \ \frac{1}{\Omega_{D-2} r^{D-2}} \frac{d\mathcal{M}(r)}{dr} \ = \ M_o \ \varphi^*(r) \ \varphi(r) \tag{48}$$

When $\varphi(r)$ is real-valued the nonlinear Klein-Gordon like equation in $D \geq 4$ is a generalization of eq-(25) given by

$$\frac{1}{r^{D-2}} \partial_r \left(\ r^{D-2} \ [1 - \frac{16\pi G_D \mathcal{M}(r)}{(D-2)\Omega_{D-2} r^{D-3}}] \ \partial_r \ (\frac{\mathcal{M}'(r)}{\Omega_{D-2} M_o r^{D-2}})^{1/2} \ \right) \ -$$

$$\frac{\lambda}{r^{D-2}} \left(\ r^{D-2} \ [1 - \frac{16\pi G_D \mathcal{M}(r)}{(D-2)\Omega_{D-2} r^{D-3}}]^{-1} \ (\frac{\mathcal{M}'(r)}{\Omega_{D-2} M_o r^{D-2}})^{1/2} \ \right) \ - \ M_o^2 \ (\frac{\mathcal{M}'(r)}{\Omega_{D-2} M_o r^{D-2}})^{1/2} \ = \ 0 \tag{49}$$

with $\mathcal{M}'(r) = \frac{d\mathcal{M}(r)}{dr}$. When $D > 4$, no longer one has that $\mathcal{M}(r) \sim r^{D-3}$ is a solution to eq-(49). The four-dimensional spacetime case was special.

In [17] we found that in certain physical scenarios Bohm's quantum potential coincided with the gravitational potential energy and that a notion of classical/quantum duality existed in the Quantum Hamilton Jacobi equation casting further light into the deep interplay between gravity and quantum mechanics. Related to the connection between Bohm's quantum potential and the gravitational potential energy is the nonlinear and novel Bohm-Poisson equation proposed by us in [16]

$$\nabla^2 Q = 4\pi G m \rho \ \Rightarrow \ - \frac{\hbar^2}{2m} \nabla^2 (\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}) \ = \ 4\pi G m \rho \tag{50}$$

where we have explicitly written the $\hbar^2$ factor in (50) (that was set to unity) for convenience.

The physical motivation behind (50) is that the laws of Physics should themselves determine the distribution density ρ of matter. It has solutions leading to repulsive gravitational behavior because eq-(50) is invariant under the transformations $G \to -G$ and $\rho \to -\rho$. It is straightforward to verify that a spherically symmetric solution to eq-(50) in a three-dim spatially flat background is given by $\rho = \frac{A}{r^4}, A = -\frac{\hbar^2}{2\pi G m^2} < 0$. Consequently $-\rho > 0$ is a valid positive- definite solution to the Bohm-Poisson equation associated to a negative gravitational coupling $-G < 0$ and which is tantamount to repulsive gravity.

The Bohm-Poisson equation was extended to the relativistic regime in [16] . Two specific solutions to the Relativistic Bohm-Poisson equation (associated to a real scalar field) were provided encoding the repulsive nature of dark energy. One solution leads to an exact cancellation of the cosmological constant, but an expanding decelerating cosmos; while the other solution leads to an exponential accelerated cosmos consistent with a de Sitter phase, and whose extremely small cosmological constant is $\Lambda = \frac{3}{R_H^2}$, consistent with current observations.

A relativistic extension of the Bohm-Poisson equation (50) can be chosen to be defined in terms of the D'Alambertian operator, and a proper mass density $\Omega(x^\mu)$ of physical dimensions $(mass/L^4) = L^{-5}$, such that $m = \int d^4x \sqrt{|g|}\Omega$, and given by an equation of the following form

$$\Box \left(\frac{\Box(\sqrt{\Omega})}{\sqrt{\Omega}} \right) \ = \ - \ 4\pi G m \Omega \tag{51}$$

Given the relativistic wave-function $\Psi(x^\mu)$ studied in the previous section, of physical dimensions L^{-2}, on dimensional grounds one can relate Ω to Ψ by setting $G m \Omega = \Psi^* \Psi$, so that eq-(51) can be rewritten as

$$\Box \left(\frac{\Box(\sqrt{\Psi^*\Psi})}{\sqrt{\Psi^*\Psi}} \right) \ = \ - \ 4\pi \Psi^* \Psi \tag{52}$$

The left and right hand side of (52) both have units of L^{-4} without having to introduce explicit $\hbar, c$ factors.

The deep question of whether or not Bohmian mechanics can be be made relativistic was studied in [18]. In relativistic space-time, Bohmian theories can be formulated by introducing a privileged foliation of space-time. The introduction of such a foliation - as an extra absolute space-time structure - would seem to imply a clear violation of Lorentz invariance, and thus a conflict with fundamental relativity. The authors [18] considered the possibility that, instead of positing it as an extra structure, the required foliation could be covariantly determined by the wave function. This allowed for the formulation of Bohmian theories that seem to qualify as fundamentally Lorentz invariant. The authors [18] concluded with some discussion of whether or not they might also qualify as fundamentally relativistic.

A stationary solution to (52) is of the form $\Psi(x^\mu) = \psi(\vec{x})\xi_o e^{i\omega t}$. After inserting it into into eq-(52), the temporal dependence decouples and it leads to

$$\nabla^2 \left(\frac{\nabla^2(\sqrt{\psi^*(\vec{x})\psi(\vec{x})})}{\sqrt{\psi^*(\vec{x})\psi(\vec{x})}} \right) = - 4\pi|\xi_o|^2\psi^*(\vec{x})\psi(\vec{x}) \quad (53)$$

which has the same functional form as the non-relativistic Bohm-Poisson equation (50). $|\xi_o|^2$ has dimensions of L^{-1}, and $\psi^*(\vec{x})\psi(\vec{x})$ have dimensions of L^{-3} so the overall dimension is L^{-4} that match the dimensions of the left-hand side (53).

If $\Psi(x^\mu)$ is real-valued, $\Psi^* = \Psi$, then eq-(52) becomes

$$\Box \left(\frac{\Box\Psi}{\Psi} \right) = - 4\pi\Psi^2 \quad (54)$$

If $\Box\Psi = m^2\Psi$ the left-hand side of (54) turns out to be zero and which does not match the right-hand side. Therefore, the solutions to equations of the form given in eq-(18) are not solutions to eq-(54).

To conclude, we should add that the Geometrization of Quantum Mechanics within the context of Finsler Gravity and Phase Spaces can be found in [19], [20]. This approach is different than the one presented here.

Acknowledgments

We are indebted to M. Bowers for assistance.

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