Deriving curves from points in the Cartesian plane

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Abstract

We derive curves from predetermined points in the Cartesian plane and obtain the elliptic.

1 Glossary

$a \in A$: a is an element of the set A .

BSD: Birch and Swinnerton-Dyer .

$\mathbb{C}$: the set of complex numbers .

CP: Cartesian product .

$\det$: determinant .

EC: elliptic curve .

EF: elliptic function .

EI: elliptic integral .

EPS: encapsulated PostScript .

FLT: Fermat's Last Theorem .

I_n: $n \times n$ identity matrix .

LHS: left-hand side .

MR: multiple root .

MT: multiplication table .

$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.

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O: the origin $(0, 0)$.

$\mathbb{Q}$: the set of rational numbers .

q: quaternion .

QED: *quod erat demonstrandum* .

$\mathbb{R}$: the set of real numbers .

RHS: right-hand side .

SR: simple root .

SVG: scalable vector graphics .

tr: trace .

w.l.o.g: without loss of generality .

$\mathbb{Z}$: the set of integers .

2 Introduction

An EC over $\mathbb{Q}$ is a nonsingular cubic curve in Weierstrass form, with rational coefficients [1]. One abstruse (and perhaps inexplicable) thing about it might be its curious etymology , which suggests that it has little to do with the so-called ellipse in the Cartesian plane , whereas EF's and EI's have been known to be related to the ellipses [2]. By definition, an EC must be nonsingular, that is, it doesn't have cusps, self-intersections, or isolated points. However, taking the aforementioned 'curiousness' seriously, we cannot resist raising the following question: What if we take those cusps, isolated points, and so forth into consideration in an attempt to study EC analogs? To address this question, we derive figures from points in the Cartesian plane and observe what they look like. Furthermore, we show some of them are closely related to the very ellipses.

3 Curve derivation

3.1 Predetermining points

At the outset, we consider the CP of the sets $\{a-c, a, a+c\}$ and $\{b-c, b, b+c\}$, where $a, b, c \in \mathbb{R}$, $c \neq 0$. That is, we consider

$$(a-c, b-c),\ (a, b-c),\ (a+c, b-c),$$

$$(a-c, b),\ (a, b),\ (a+c, b),$$

$$(a-c, b+c),\ (a, b+c),\ (a+c, b+c).$$

For the sake of simplicity, we set $a = b = 0$ and $c = 1$. We thus treat the CP of the sets $\{-1, 0, 1\}$ and $\{-1, 0, 1\}$, which are $(-1, -1), (0, -1), (1, -1), (-1, 0), (0, 0), (1, 0), (-1, 1), (0, 1), (1, 1)$. We regard them as nine points in the Cartesian plane as visualised below. [1]

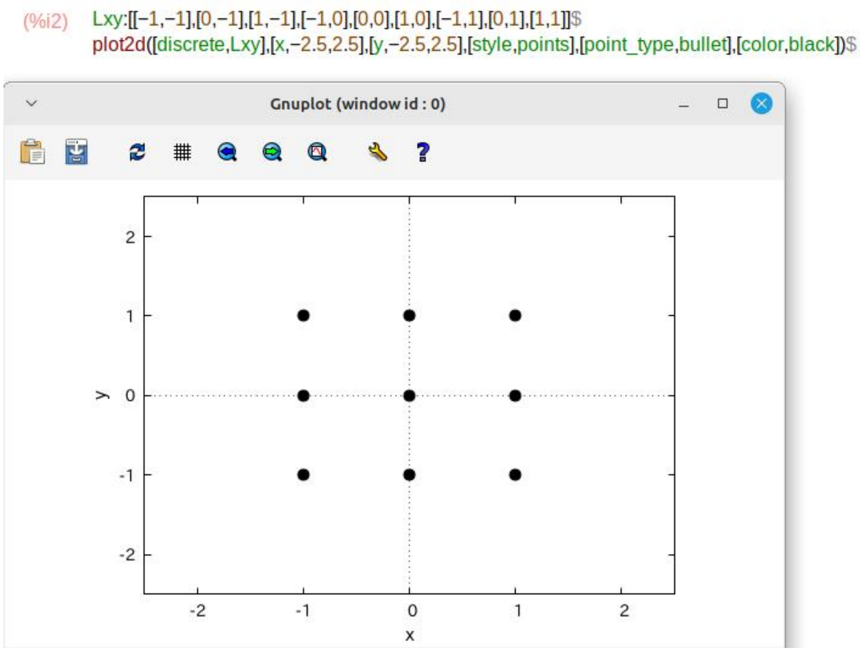

Fig. 1. Nine points which have been predetermined

3.2 Point-determined equation

Let us consider the equation

$$(y + 1)y(y - 1) = (x + 1)x(x - 1) \tag{1}$$

which reflects the aforementioned nine points.

Graphically,

[1]From now on, we use wxMaxima ver. 20.12.1 for visualisation unless otherwise specified.

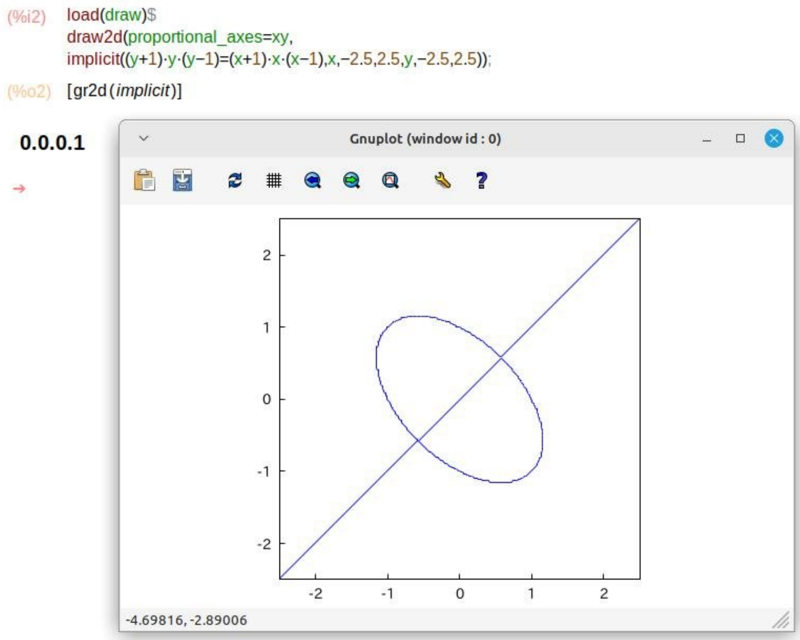

Fig. 2. (1) visualised

Since the visualised stuff in the above figure seems to resemble the Greek character ϕ , we tentatively call it "ϕ"-curve. [2] Putting Fig.'s 1 and 2 together, one gets the following.

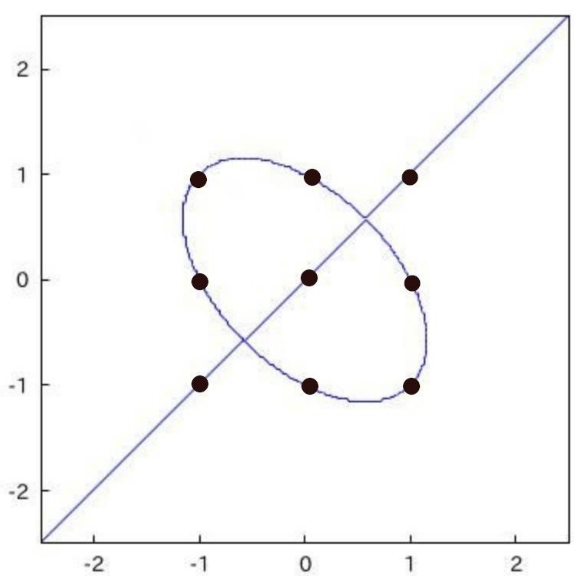

Fig. 3. Fig. 1 overlaid on Fig. 2. We later insetted the points in Fig. 1 using Pinta ver. 1.6.

[2]Some might recall "alpha" curve [3, Figure 2].

By the way, factoring $(x+1)x(x-1)-(y+1)y(y-1)$ yields $(x-y)(x^2+xy+y^2-1)$, which shows that (1) is a union of the line [3] $x-y=0$ and

$$x^2+xy+y^2-1=0. \tag{2}$$

Putting the line aside, we call (2) 'something' until we understand its geometric nature. To that end, we prove the following in advance.

PROPOSITION 3.2.1. An ellipse whose centre is O remains to be an ellipse after a rotation around O.

Proof. W.l.o.g., we may assume an ellipse before such a rotation is

$$E_1 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a>b>0.^{4} \tag{3}$$

Next, let a counterclockwise rotation by an angle θ around O be denoted by

$$R(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The foci of E_1 are $f(\sqrt{a^2-b^2},0)$ and $f'(-\sqrt{a^2-b^2},0)$. So $R(\theta)$ carries f and f' to

$$F:\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}=\begin{pmatrix} \sqrt{a^2-b^2}\cos\theta \\ \sqrt{a^2-b^2}\sin\theta \end{pmatrix}$$

and

$$F':\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sqrt{a^2-b^2} \\ 0 \end{pmatrix}=\begin{pmatrix} -\sqrt{a^2-b^2}\cos\theta \\ -\sqrt{a^2-b^2}\sin\theta \end{pmatrix},$$

respectively. Since an ellipse can be defined as the locus of points for which the sum of the distances to two given foci is constant , rewriting $\sqrt{a^2-b^2}$ as c, one has

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=\text{Constant}. \tag{4}$$

And F and F' are rewritten as $F(c\cos\theta, c\sin\theta)$ and $F'(-c\cos\theta,-c\sin\theta)$. By further rewriting $c\cos\theta$ and $c\sin\theta$ as d and e, respectively, now they are $F(d,e)$ and $F'(-d,-e)$. We compute the sum of the distances to F and F' as follows.

$$\sqrt{(x-d)^2+(y-e)^2}+\sqrt{(x+d)^2+(y+e)^2}$$

[3] The so-called line is regarded as a kind of curve.

[4] See **10.1** for why this assumption is legitimate.

$$= \sqrt{x^2 + y^2 - 2(dx + ey) + d^2 + e^2} + \sqrt{x^2 + y^2 + 2(dx + ey) + d^2 + e^2}$$

$$= \sqrt{x^2 + y^2 - 2(dx + ey) + c^2} + \sqrt{x^2 + y^2 + 2(dx + ey) + c^2}$$

$$(\because d^2 + e^2 = c^2)$$

$$= \sqrt{x^2 + y^2 - 2c(x\cos\theta + y\sin\theta) + c^2} + \sqrt{x^2 + y^2 + 2c(x\cos\theta + y\sin\theta) + c^2}$$

$$(\because d = c\cos\theta, \text{ and } e = c\sin\theta)$$

$$= \sqrt{(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 - 2c(x\cos\theta + y\sin\theta) + c^2}$$
$$+\sqrt{(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 + 2c(x\cos\theta + y\sin\theta) + c^2}.$$

Replacing $x\cos\theta + y\sin\theta$ and $-x\sin\theta + y\cos\theta$ by X and Y, respectively yields $\sqrt{X^2 + Y^2 - 2cX + c^2}$ $+\sqrt{X^2 + Y^2 + 2cX + c^2}$. After some computation, this becomes $\sqrt{(X-c)^2 + Y^2} + \sqrt{(X+c)^2 + Y^2}$, which is essentially the same as the LHS of (4). Since the RHS of (4) is constant, we can say that $\sqrt{(x-d)^2 + (y-e)^2} + \sqrt{(x+d)^2 + (y+e)^2}$ is also constant. We now notice that the replacement we used, *i.e.*,

$$\begin{cases} X = x\cos\theta + y\sin\theta, \\ Y = -x\sin\theta + y\cos\theta \end{cases}$$

can be rewritten as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R(-\theta) \begin{pmatrix} x \\ y \end{pmatrix}.$$

This eventually means that a rotation of E_1 by an angle $-\theta$ around O gave rise to another ellipse. QED.

COROLLARY 3.2.2. In the above PROPOSITION, foci remain to be foci after such a rotation.

Proof. Since we have shown $\sqrt{(x-d)^2 + (y-e)^2} + \sqrt{(x+d)^2 + (y+e)^2}$ is constant in the PROPOSITION, (d, e) and $(-d, -e)$, which come from the foci f and f', respectively are foci, too. QED.

Claim 3.2.3. 'Something' *is* an ellipse.

Proof. Consider

$$\begin{pmatrix} x \\ y \end{pmatrix} = R(\theta) \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Explicitly,

$$\begin{cases} x = X\cos\theta - Y\sin\theta, \\ y = X\sin\theta + Y\cos\theta. \end{cases}$$

Substituting these into the LHS of (2), one gets

$$(X\cos\theta - Y\sin\theta)^2 + (X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + (X\sin\theta + Y\cos\theta)^2 - 1 = 0.$$

After some computation, this becomes

$$(1+\sin\theta\cos\theta)X^2 + \cos 2\theta XY + (1-\sin\theta\cos\theta)Y^2 - 1 = 0. \tag{5}$$

If we set

$$2\theta = n\pi + \frac{\pi}{2}, \qquad n \in \mathbb{Z} \tag{6}$$

in (5), since $\cos(n\pi + \frac{\pi}{2}) = \cos(n\pi)\cos(\frac{\pi}{2}) - \sin(n\pi)\sin(\frac{\pi}{2})$ [5] $= \cos(n\pi)\cdot 0 - 0\cdot\sin(\frac{\pi}{2}) = 0$, (5) becomes

$$(1+\sin\theta\cos\theta)X^2 + (1-\sin\theta\cos\theta)Y^2 = 1. \tag{7}$$

We choose to set $n = -1$ in (6). Then, $\theta = (-\pi + \frac{\pi}{2})/2 = -\frac{\pi}{4}$, and (7) becomes

$$\{1 + (-\frac{1}{\sqrt{2}})\cdot\frac{1}{\sqrt{2}}\}X^2 + \{1 - (-\frac{1}{\sqrt{2}})\cdot\frac{1}{\sqrt{2}}\}Y^2 = 1,$$

which is

$$\frac{X^2}{2} + \frac{Y^2}{\frac{2}{3}} = 1, \tag{8}$$

an ellipse. Conceptually, we have that

$$(X, Y) \xrightarrow{\text{rotation by an angle } \theta} (x, y).$$

Concretely, since we set $\theta = -\frac{\pi}{4}$, we have

$$(X, Y) \xrightarrow{\text{rotation by an angle } -\frac{\pi}{4}} (x, y),$$

that is,

$$(8) \xrightarrow{\text{rotation by an angle } -\frac{\pi}{4}} x^2 + xy + y^2 = 1 \text{ (or `something')}.$$

[5] Here we used the formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Since (8) is an ellipse whose centre lies in O, it follows from PROPOSITION 3.2.1 that (2) (or 'something') is an ellipse, too. QED.

Now that we have proven that 'something' (or (2)) is an ellipse, we can say the object visualised in Fig. 2 is a union of a line and an ellipse. So we denote (2) (or 'something') by E_2.[6]

Remark 3.2.4. However, this proof might have just corroborated what is intuitively clear.[7]

Remark 3.2.5. $\det$-based classification of conic sections is known, which is also suitable for saying (2) (or 'something') is an ellipse.

4 Computing *SING* $[4]$

We compute the *SING* of (1), for that matter. Consider

$$\phi = (x+1)x(x-1) - (y+1)y(y-1) = 0.$$

So

$$\begin{aligned}\tfrac{d\phi}{dx} &= \tfrac{d}{dx}\{(x+1)x(x-1) - (y+1)y(y-1)\} = \tfrac{d}{dx}(x+1)x(x-1) + \tfrac{d}{dx}\{-(y+1)y(y-1)\} \\ &= 3x^2 - 1 + \tfrac{dy}{dx} \cdot \tfrac{d}{dy}(-y^3 + y) = 3x^2 - 1 + \tfrac{dy}{dx}(-3y^2 + 1).\end{aligned}$$

We thus get the 1-form $\omega = d\phi = (3x^2 - 1)dx - (3y^2 - 1)dy$. In order to compute *SING*, we set

$$\begin{cases} 3x^2 - 1 = 0, \\ 3y^2 - 1 = 0. \end{cases}$$

We solve the above to get the following four points.

$$S1(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}), \quad S2(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}), \quad S3(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}), \quad S4(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}).$$

These are the *SING*'s of (1), and we visualise them:

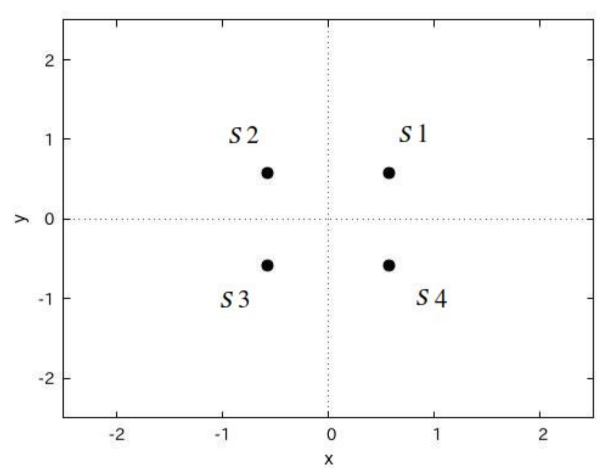

Fig. 4. Four *SING*'s of (1)

[6] *Cf.* (3).

[7] For a (rather) intuitive proof, see **10.2**.

We put Fig.'s 3 and 4 together like the following.

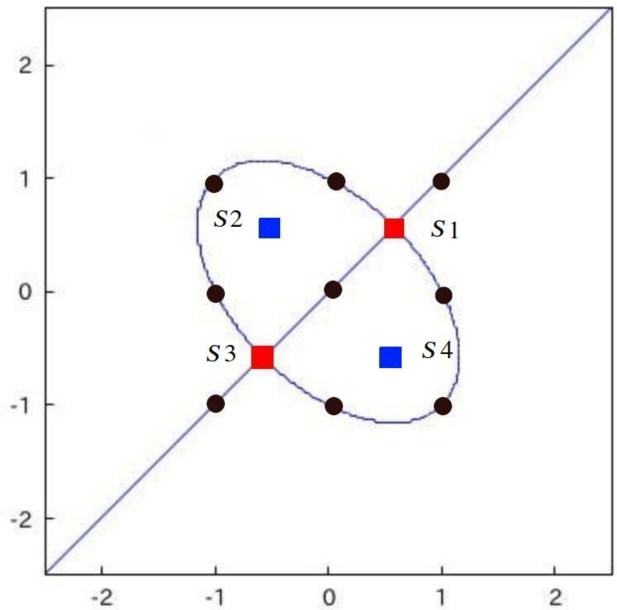

Fig. 5. Fig. 4 overlaid on Fig. 3. We later insetted *SING*'s in Fig. 4 like Fig. 3. [8]

We should like to make some claims about the *SING*'s in the above figure.

Claim 4.1. *SING*'s $S2$ and $S4$ don't coincide with the foci of E_2.

Proof. In what follows, double-signs correspond. Foci of (8) are $f(\frac{2}{\sqrt{3}}, 0)$ and $f'(-\frac{2}{\sqrt{3}}, 0)$. By the way, we have

$$\begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} \pm\frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \pm\frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} \pm\frac{\sqrt{2}}{\sqrt{3}} \\ \mp\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

And using COROLLARY 3.2.2, one can say $(\pm\frac{\sqrt{2}}{\sqrt{3}}, \mp\frac{\sqrt{2}}{\sqrt{3}})$ are foci of (2). However, they are different from $S2$ and $S4$. QED.

Claim 4.2. *SING*'s $S1$ and $S3$ coincide with the co-vertices of E_2.

Proof. The co-vertices of (8) are $(0, \frac{\sqrt{2}}{\sqrt{3}})$ and $(0, -\frac{\sqrt{2}}{\sqrt{3}})$. And we have

[8] Since there seem to be two kinds of *SING*'s, we indicate them in either red or blue.

$$\begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} 0 \\ \pm\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \pm\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix},$$

where double-signs correspond. By the way, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ coincide with $S1$ and $S3$, respectively. QED.

5 Some generalisation

Since we have dealt with (rather) simple figures, we wonder if we can generalise them to some extent by *e.g.*, replacing O by an arbitrary point (α, β). Specifically, we consider the CP of the sets

$$\{\alpha - 1, \alpha, \alpha + 1\} \text{ and } \{\beta - 1, \beta, \beta + 1\},$$

where we assume $0 < \alpha < \beta$ [9].

Graphically,

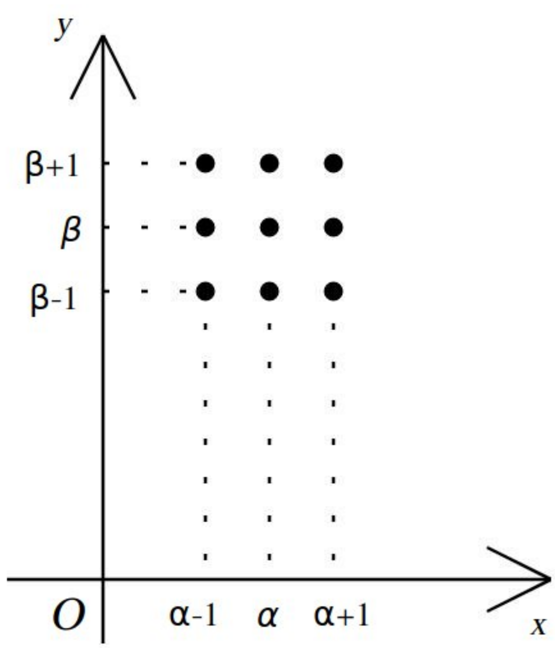

Fig. 6. Fig. 1 slightly generalised [10]

We consider

[9] *Cf.* **3.1**.

[10] An SVG file was converted to an EPS file, which was included in a usual LaTeX manner .

$$\phi = (x-\alpha+1)(x-\alpha)(x-\alpha-1) - (y-\beta+1)(y-\beta)(y-\beta-1) = 0$$

and compute

$$\begin{aligned}
\tfrac{d\phi}{dx} &= \tfrac{d}{dx}\{(x-\alpha+1)(x-\alpha)(x-\alpha-1) - (y-\beta+1)(y-\beta)(y-\beta-1)\} \\
&= \tfrac{d}{dx}(x-\alpha+1)(x-\alpha)(x-\alpha-1) + \tfrac{d}{dx}\{-(y-\beta+1)(y-\beta)(y-\beta-1)\} \\
&= \tfrac{d}{dx}\{x^3 - 3\alpha x^2 + (3\alpha^2-1)x - \alpha^3 + \alpha\} + \tfrac{dy}{dx}\cdot\tfrac{d}{dy}\{-y^3 + 3\beta y^2 - (3\beta^2-1)y + \beta^3 - \beta\} \\
&= 3x^2 - 6\alpha x + 3\alpha^2 - 1 + \tfrac{dy}{dx}\{-3y^2 + 6\beta y - (3\beta^2-1)\}.
\end{aligned}$$

So we write

$$\omega = d\phi = (3x^2 - 6\alpha x + 3\alpha^2 - 1)dx - (3y^2 - 6\beta y + 3\beta^2 - 1)dy$$

and solve

$$\begin{cases} 3x^2 - 6\alpha x + 3\alpha^2 - 1 = 0, \\ 3y^2 - 6\beta y + 3\beta^2 - 1 = 0 \end{cases}$$

to get the *SING*'s

$$(x,y) = (\alpha + \frac{1}{\sqrt{3}}, \beta + \frac{1}{\sqrt{3}}),\ (\alpha - \frac{1}{\sqrt{3}}, \beta + \frac{1}{\sqrt{3}}),\ (\alpha - \frac{1}{\sqrt{3}}, \beta - \frac{1}{\sqrt{3}}),\ (\alpha + \frac{1}{\sqrt{3}}, \beta - \frac{1}{\sqrt{3}}). \quad (9)$$

Remark 5.1. Setting $\alpha = \beta = 0$ in (9) yields $S1 - S4$. Regarding them as special cases of (9), we can say we have generalised $S1 - S4$ at least to some extent.

By the way, since there are four points in Fig. 4, it seems natural one should raise the following question.

Question 5.2. Is it still possible to derive curves from < 9 predetermined points?

We will try to answer this in the following section.

6 Starting from < 9 points

6.1 Getting another "ϕ"-curve

We try allowing for the existence of MR's. For example, we consider

$$y(y-1)^2 = x(x-1)^2 \quad (10)$$

and regard $x = 1$ and $y = 1$ as MR's in the RHS and LHS of (10), respectively. Then, we visualise (10) to get "ϕ"-curve again: [11]

[11] *Cf.* Fig. 2.

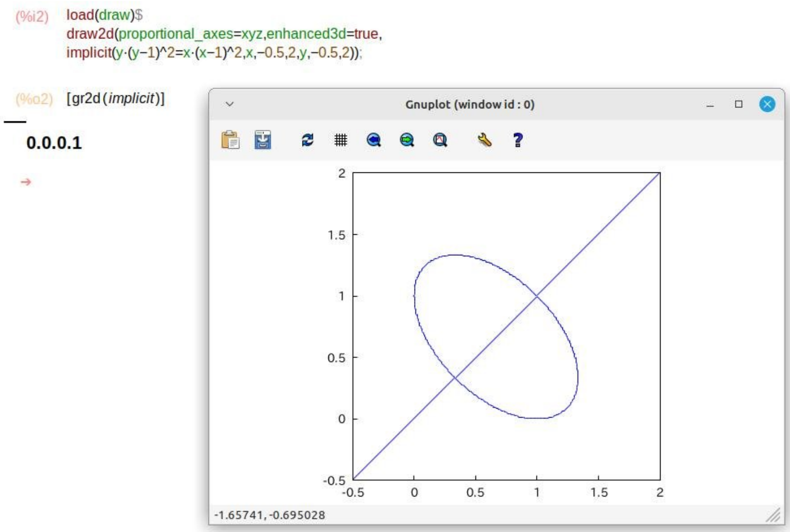

Fig. 7. Another “ϕ”-curve

So our response to *Question* 5.2 is

Answer 6.1.1. If MR’s are allowed for and counted as one root, we can derive a curve from < 9 points. [12]

Talking of *SING*’s, we consider

$$\phi = x(x-1)^2 - y(y-1)^2 = 0$$

and compute

$$\begin{aligned}\tfrac{d\phi}{dx} &= \tfrac{d}{dx}\{x(x-1)^2 - y(y-1)^2\} = 3x^2 - 4x + 1 + \tfrac{dy}{dx}\cdot\tfrac{d}{dy}\{-y(y-1)^2\}\\ &= 3x^2 - 4x + 1 + \tfrac{dy}{dx}(-3y^2 + 4y - 1).\end{aligned}$$

So we get

$$d\phi = (3x^2 - 4x + 1)dx - (3y^2 - 4y + 1)dy,$$

and in order to compute *SING*, we solve

$$\begin{cases} 3x^2 - 4x + 1 = 0,\\ 3y^2 - 4y + 1 = 0\end{cases}$$

to get

$$(x, y) = (\tfrac{1}{3}, \tfrac{1}{3}),\ (\tfrac{1}{3}, 1),\ (1, \tfrac{1}{3}),\ (1, 1).$$

We overlay these points on Fig. 7:

[12] Specifically, we started from the four points $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.

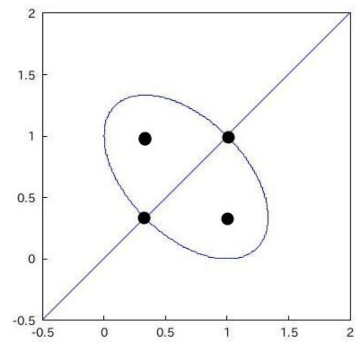

Fig. 8. Another "ϕ"-curve and its *SING*'s. They were put together like Fig. 5.

6.2 Getting two intersecting lines

We now try to derive a curve from < 9 distinct points. [13] For example, we visualise

$$y(y-1) = x(x-1) \tag{11}$$

that passes through the four points $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ as follows.

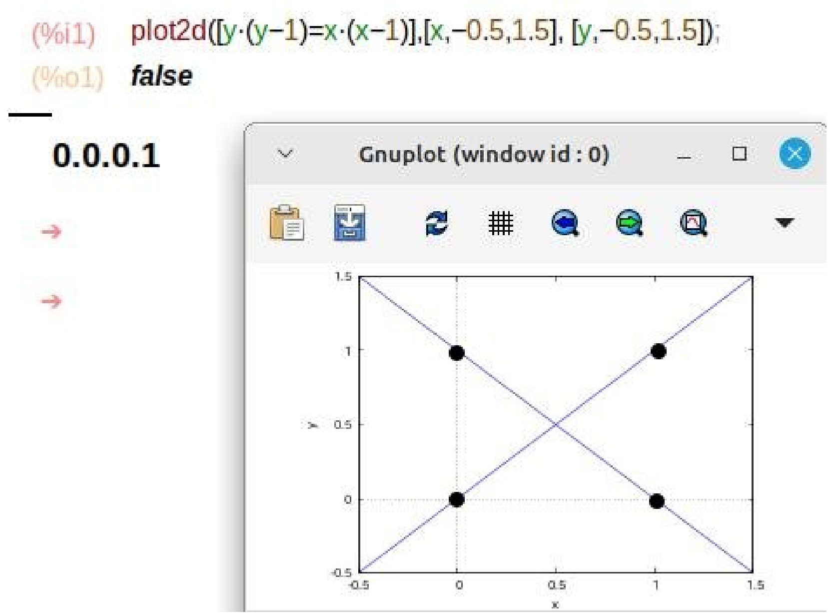

Fig. 9. Two intersecting lines. Four predetermined points were later insetted like Fig. 3.

[13]By 'distinct', we mean that we allow for SR's only.

We then consider

$$\phi = x(x-1) - y(y-1) = 0$$

and compute

$$\frac{d\phi}{dx} = \frac{d}{dx}\{x(x-1) - y(y-1)\} = 2x - 1 + \frac{dy}{dx} \cdot \frac{d}{dy}\{-y(y-1)\} = 2x - 1 + \frac{dy}{dx}(-2y+1).$$

So we get

$$d\phi = (2x-1)dx - (2y-1)dy.$$

We solve

$$\begin{cases} 2x - 1 = 0, \\ -2y + 1 = 0 \end{cases}$$

to get the *SING* of (11), $(\frac{1}{2}, \frac{1}{2})$, which is overlaid on Fig. 9 as follows.

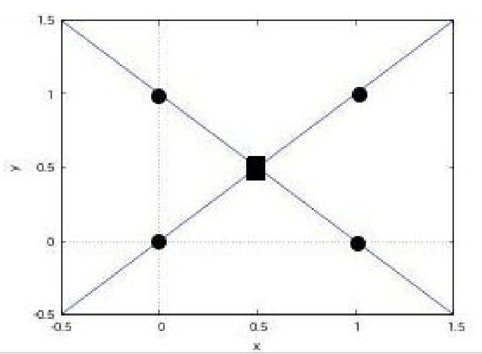

Fig. 10. Two intersecting lines and their *SING*. They were put together like Fig. 8, *SING* being indicated by a solid black square.

7 On 'higher-dimensionalisation'

In this section, we try to 'higher-dimensionalise' some figures we have so far mentioned.

7.1 Regarding E_2 as originally three-dimensional

We imagine the sphere

$$x^2 + y^2 + z^2 = r^2.$$

Then, we replace z by $ax + by$, $a, b \in \mathbb{R}$ to get

$$x^2 + y^2 + (ax + by)^2 = r^2.$$

Expanding its LHS, one gets

$$(a^2 + 1)x^2 + 2abxy + (b^2 + 1)y^2 = r^2,$$

and setting $a = b = \pm 1$, where double-signs correspond, and $r = \pm\sqrt{2}$ yields

$$2x^2 + 2xy + 2y^2 = 2,$$

which is essentially the same as (2). So the section of the sphere by the planes $z = x + y$ or $z = -x - y$ leads to E_2, which we thus regard as derivable from the sphere, a three-dimensional object. In this way, E_2 has been related to something 'higher-dimensional' and 'lifted up' by one dimension.

7.2 A 'trinionic' representation of (1)

As another way for 'higher-dimensionalisation', we try representing (1) by 'trinions' (t_r's) [5]. We first write

$$x = x_1 + x_2 i + x_3 j, \quad y = y_1 + y_2 i + y_3 j, \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}. \tag{12}$$

We then get

$$x^3 = x_1^3 + 3x_1^2(x_2 i + x_3 j), \quad y^3 = y_1^3 + 3y_1^2(y_2 i + y_3 j) \tag{13}$$

after some calculations. [14]

On the other hand, after some expansion, (1) becomes

$$x^3 - x = y^3 - y. \tag{14}$$

Substituting (12) and (13) into (14), we get

$$x_1^3 + 3x_1^2(x_2 i + x_3 j) - (x_1 + x_2 i + x_3 j) = y_1^3 + 3y_1^2(y_2 i + y_3 j) - (y_1 + y_2 i + y_3 j).$$

After some computation, one gets

$$x_1^3 - x_1 + x_2(3x_1^2 - 1)i + x_3(3x_1^2 - 1)j = y_1^3 - y_1 + y_2(3y_1^2 - 1)i + y_3(3y_1^2 - 1)j.$$

[14] For MT required for this kind of calculation, see [5, **Table 1**].

So we have

$$\begin{cases} x_1^3 - x_1 = y_1^3 - y_1, & (15)\\ x_2(3x_1^2 - 1) = y_2(3y_1^2 - 1), & (16)\\ x_3(3x_1^2 - 1) = y_3(3y_1^2 - 1). & (17) \end{cases}$$

Since (15) is essentially the same as (1), we ignore it. By considering the coordinate (x_1, x_2, y_1, y_2), one can regard (16) as something four-dimensional. (17) can also be regarded as something four-dimensional by thinking of the coordinate (x_1, x_3, y_1, y_3).

N.B. In what follows, '$\imath$' in q needs to be differentiated from 'i' in t_r.

8 3-parameter representation of a rotation matrix in three dimension

Having touched upon something four-dimensional in the preceding subsection, we recall $\mathbb{R}^4$ and $q = a + b\imath + cj + dk$, where $a, b, c, d \in \mathbb{R}$, in particular, unit q, where $a^2 + b^2 + c^2 + d^2 = 1$. As an example in which unit q plays a non-negligible role, we mention

$$R(a,b,c,d) = \left(\begin{array}{ccc} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2+c^2-b^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2+d^2-b^2-c^2 \end{array} \right).$$

We rewrite this 3×3 matrix using the following replacement

$$\begin{cases} a = \cos\alpha, \\ b = \sin\alpha\cos\beta, \\ c = \sin\alpha\sin\beta\cos\gamma, \\ d = \sin\alpha\sin\beta\sin\gamma, \end{cases}$$

where $a^2 + b^2 + c^2 + d^2 = 1$ holds, as

$$S(\alpha,\beta,\gamma)$$
$$= \begin{pmatrix} 1-2(\sin\alpha\sin\beta)^2 & 2\sin\alpha\sin\beta(\sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin\gamma) & 2\sin\alpha\sin\beta(\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma) \\ 2\sin\alpha\sin\beta(\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma) & \cos 2\alpha+2(\sin\alpha\sin\beta\cos\gamma)^2 & 2\sin\alpha(\sin\alpha\sin^2\beta\cos\gamma\sin\gamma-\cos\alpha\cos\beta) \\ 2\sin\alpha\sin\beta(\sin\alpha\cos\beta\sin\gamma-\cos\alpha\cos\gamma) & 2\sin\alpha(\sin\alpha\sin^2\beta\cos\gamma\sin\gamma+\cos\alpha\cos\beta) & \cos 2\alpha+2(\sin\alpha\sin\beta\sin\gamma)^2 \end{pmatrix}.$$

We make some claims.

Claim 8.1. Global maxima and minima of $\mathrm{tr}(S(\alpha,\beta,\gamma))$ are 3 and -1, respectively.

Proof. $\mathrm{tr}(S(\alpha,\beta,\gamma)) = 1-2(\sin\alpha\sin\beta)^2+\cos 2\alpha+2(\sin\alpha\sin\beta\cos\gamma)^2+\cos 2\alpha+2(\sin\alpha\sin\beta\sin\gamma)^2$ $= 2\cos 2\alpha + 1$. It follows from $-2 \le 2\cos 2\alpha \le 2$ that $-1 \le 2\cos 2\alpha + 1 \le 3$. QED.

Now let

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \ R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim 8.2. $S(\pi,\pi,\pi) = R_x(2\pi) = R_y(2\pi) = R_z(2\pi) = I_3$.

Proof. A straightforward computation. QED.

Remark 8.3. We notice $\det R_x(\phi) = \det R_y(\phi) = \det R_z(\phi) = 1$, and $\mathrm{tr}(R_x(\phi)) = \mathrm{tr}(R_y(\phi)) = \mathrm{tr}(R_z(\phi)) = 2\cos\phi + 1$. So if $\phi = 2\pi n$, $n \in \mathbb{Z}$, $\mathrm{tr}(R_x(\phi)) = \mathrm{tr}(R_y(\phi)) = \mathrm{tr}(R_z(\phi)) = 3$. Hence, $R_x(\phi), R_y(\phi), R_z(\phi)$ can be ST_3, or 3×3 'special trace matrix' [6].

9 Discussion

As suggested in footnote 8, we group *SING*'s in Fig. 5 into two **IN**'s (solid blue squares) and two **ON**'s (solid red squares) in terms of *SING* classification [4, **Table**]. So (1) is classified into the categories **IN** and **ON** simultaneously. To the best of our knowledge, this is the first case of such simultaneity. Moreover, since for example, both $S1$ and $S3$ in Fig. 5 belong to the same category **ON**, it was shown that multiple *SING*'s can belong to one category. However, in Fig. 10, the sole *SING* is classified into the category **ON**. To sum up, the so-called 'one-to-one correspondence' does not always hold for *SING*'s.

As for 'higher-dimensionalisation', it might be of some interest, although it is known that 'two-dimensionalisation' of something three-dimensional is possible [7, FIG. 1.82] and that EC's over $\mathbb{C}$ are identified with complex tori of dimension one [8].

Regarding 3-parameter representation of a rotation matrix, $\mathrm{tr}(S(\alpha,\beta,\gamma))$ includes only α as its variable, whereas $\mathrm{tr}(R(a,b,c,d)) = 3a^2 - b^2 - c^2 - d^2$, there being four variables. So we can say such a representation resulted in the reduction of variables.

By the way, what are *explicitly* elliptic in the Cartesian plane? Thinkable are

- the so-called ellipse [9];
- section of sphere coming from the elimination of 'z' [15];
- some Lissajous curves [10];

etc. Unfortunately, EC's seem irrelevant after all. That said, we notice that an equation we dealt with has something to do with an EC. Schematically,

$$y^3 - y = x^3 - x \qquad (14)$$

$$\downarrow \qquad \text{Add } \tfrac{1}{4} \text{ to both sides.}$$

[15] See **7.1**.

$$
\begin{array}{rl}
y^3 - y + \frac{1}{4} = x^3 - x + \frac{1}{4}. & \\
\downarrow & \text{Replace } y^3 \text{ by } y^2. \\
y^2 - y + \frac{1}{4} = x^3 - x + \frac{1}{4}. & \\
\downarrow & \text{Square-complete the LHS.} \\
(y - \frac{1}{2})^2 = x^3 - x + \frac{1}{4}. & \\
\uparrow & \text{A translation by } \frac{1}{2}. \\
y^2 = x^3 - x + \frac{1}{4} &
\end{array}
$$

Hence, (14) is not unrelated to the EC $y^2 = x^3 - x + \frac{1}{4}$ [16]. That a slight change in the exponent results in a non-negligible change is not unusual. For example, $x^3 + y^3 = z^3$ has no solution in $\mathbb{N}$ due to FLT, whereas $x^3 + y^3 = z^2$ *does* have solutions such as $(1, 2, 3)$, $(2, 2, 4)$, and so on. Furthermore, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is elementary, whereas $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ is not $[11]$.

Finally, we wonder if it is worth trying to

- get a deeper understanding of the BSD conjecture, to which an EC is relevant $[12]$;
- apply *e.g.*, Fig. 6 to microarray data to get some biophysical insights;

etc.

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10 Appendix

10.1 Why is there no loss of generality in the assumption of the proof of PROPOSITION 3.2.1?

We deal with cases where each ellipse differs from (3), but its centre is O.

Case 1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b > a > 0.$
Case 2. *Case* 1 is rotated by an angle θ, $0 < \theta < \frac{\pi}{2}$, around O.
Case 3. *Case* 1 is rotated by an angle θ, $0 < \theta < -\frac{\pi}{2}$, around O.

In *Case* 1, replacing x and y by Y and X, respectively yields $\frac{X^2}{b^2} + \frac{Y^2}{a^2}$, $b > a > 0$, which is essentially the same as (3). In *Case* 2, if one replaces its semi-major axis and semi-minor axis by x-axis and y-axis, respectively, it can be regarded as essentially the same as (3), which is also the case with *Case* 3. Hence, all thinkable kinds of ellipses are reduced to (3), which is why we regard (3) as sufficiently general.

10.2 A (rather) intuitive proof of *Claim* 3.2.3

We intuitively substitute

$$\begin{cases} x = \frac{X+Y}{\sqrt{2}}, & (18) \\ y = \frac{-X+Y}{\sqrt{2}} & (19) \end{cases}$$

into the LHS of (2) to get

$$(\tfrac{X+Y}{\sqrt{2}})^2 + \tfrac{X+Y}{\sqrt{2}} \cdot \tfrac{X-Y}{\sqrt{2}} + (\tfrac{X-Y}{\sqrt{2}})^2 - 1 = 0.$$

After some computation, the above becomes

$$\frac{X^2}{\frac{2}{3}} + \frac{Y^2}{2} = 1,$$

which is an ellipse. By the way, (18) and (19) can be rewritten as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

which is a clockwise rotation by an angle $\frac{\pi}{4}$ around O. And since it is intuitively clear that an ellipse whose centre is O becomes another ellipse after such a rotation, (2) is an ellipse, too. QED.

10.3 On 'latent' ellipse

If we are allowed to replace the constant a_2 in

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad [1]$$

by $a_2 - a_5$, we can rewrite the above as

$$y^2 + a_1xy + a_3y = x^3 + (a_2 - a_5)x^2 + a_4x + a_6$$

and get

$$y^2 + a_1xy + a_3y + a_5x^2 = x^3 + a_2x^2 + a_4x + a_6.$$

The LHS of this is an ellipse if discriminant $a_1^2 - 4a_5 < 0$.

Example 10.3.1. $a_1 = a_3 = a_5 = 1$. That is, $y^2 + xy + x^2 + y$. Equating this with 0, one sees an ellipse as shown below.