Fundamental Physics as the General Solution to a Maximization Problem on the Shannon Entropy of All Measurements

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Abstract

We propose a novel approach to quantum theory construction that involves solving a maximization problem on the Shannon entropy of all possible measurements of a system, relative to its initial preparation. This maximization problem is additionally constrained by a phase condition that vanishes under measurements. Specifically, enforcing a vanishing U(1)-valued phase constraint leads to standard quantum mechanics, while a vanishing Spin\(^{(3,1)}\)-valued phase constraint extends the theory to relativistic quantum mechanics and to quantum gravity. The latter scenario is found to incorporate the SU(3) × SU(2) × U(1) symmetries of the Standard Model as invariants of the probability measure itself, and to construct the metric tensor as an operator via a double-copy mechanism applied to the Dirac current. Significantly, this solution is consistent exclusively with a 3+1-dimensional spacetime configuration—other dimensional settings are shown to lead to fundamental obstructions. This framework seamlessly integrates fundamental concepts from quantum mechanics, relativistic quantum mechanics, quantum gravity, dimensional specificity of spacetime, and particle physics symmetries into a unified entropy maximization problem constrained by a vanishing phase.

1 Introduction

The canonical formalism of quantum mechanics (QM) is based on five principal axioms[1, 2]:

QM Axiom 1 of 5 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system’s state represented by a ray in this space.

QM Axiom 2 of 5 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

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QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system’s total energy.

QM Axiom 4 of 5 **Measurement:** The act of measuring an observable results in the system’s transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

QM Axiom 5 of 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector’s projection onto the relevant eigenstate.

Contrastingly, statistical mechanics (SM), the other statistical pillar of physics, derives its probability measure through entropy maximization, constrained by the following expression:

SM Constraint 1 of 1: **Average Energy Constraint:** The average of energy measurements of a system at thermodynamic equilibrium converge to a specific value (\( \bar{E} \)):

\[
\bar{E} = \sum_{q \in Q} \rho(q) E(q)
\]

(1)

To maximize entropy while satisfying this constraint, the theory uses a Lagrange multiplier approach.

**Definition 1** (Fundamental Lagrange Multiplier Equation of SM).

\[
\mathcal{L}(\rho, \lambda, \beta) = -k_B \sum_{q \in Q} \rho(q) \ln \rho(q) + \lambda \left( 1 - \sum_{q \in Q} \rho(q) \right) + \beta \left( \bar{E} - \sum_{q \in Q} \rho(q) E(q) \right)
\]

(2)

where \( \lambda \) and \( \beta \) are the Lagrange multipliers.

**Theorem 1** (Gibbs Measure). The solution to the Lagrange multiplier equation of SM is the Gibbs measure.

\[
\rho(q) = \frac{1}{\sum_{r \in Q} \exp(-\beta E(r))} \exp(-\beta E(q))
\]

(3)

**Proof.** This is an well-known result by E. T. Jaynes [3, 4]. As a convenience, we replicate the proof in Annex A. \( \square \)

As evident from E. T. Jaynes’ methodological innovation, SM relies on a single constraint related to the nature of the measurements under consideration, which allows the formulation of an optimization problem sufficient to derive the
relevant probability measure. This is an exceptionally parsimonious formulation of a physical theory.

We propose a generalization of E. T. Jaynes’ approach to the realms of Quantum Mechanics (QM), relativistic Quantum Mechanics (RQM) and Quantum Gravity (QG). For each domain, we will introduce a single constraint related to measurements, formulate a corresponding entropy maximization problem, and present a main theorem that encapsulates the theory. This formulation reduces fundamental physics to its most parsimonious expression, deriving the core theories as optimal solutions to a well-defined entropy maximization problem.

1.1 Quantum Mechanics

To reformulate QM as the solution to an entropy maximization problem, we propose the following constraint:

QM Constraint 1 of 1 Vanishing Complex-Phase: Quantum measurements admit a vanishing complex phase. The constraint is:

\[ 0 = \text{tr} \sum_{q \in Q} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \]  \hspace{1cm} (4)

Here, the matrix representation engenders the complex phase, and the trace will cause it to vanish under measurement.

It associates to the following equation:

**Definition 2** (Fundamental Lagrange Multiplier Equation of QM).

\[ \mathcal{L}(\rho, \lambda, \tau) = - \sum_{q \in Q} \rho(q) \ln \frac{\rho(q)}{\rho_0(q)} + \lambda \left( 1 - \sum_{q \in Q} \rho(q) \right) + \tau \left( - \text{tr} \sum_{q \in Q} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) \]  \hspace{1cm} (5)

where \( \lambda \) and \( \tau \) are the Lagrange multipliers.

The relative Shannon entropy[5, 6] is utilized because we are solving for the least biased theory that connects an initial preparation \( p(q) \) to its final measurement \( \rho(q) \).

**Theorem 2.** The least biased probability measure that connects an initial preparation \( p(q) \) to its final measurement \( \rho(q) \), under the constraint of the vanishing complex-phase, is:

\[ \rho(q) = \frac{1}{\sum_{r \in Q} p(r) \| \exp(-itE(r)/\hbar) \| \| \exp(-itE(q)/\hbar) \|} \begin{bmatrix} \rho(q) \\ p(q) \end{bmatrix} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \]  \hspace{1cm} (6)

where we have defined \( \tau = t/\hbar \) (analogous to \( \beta = 1/(k_B T) \) in SM).
The proof of this theorem will be presented in the results section. We will show that this solution entails the five axioms of QM, which are now promoted to theorems, yielding a parsimonious formulation of QM.

1.2 Relativistic Quantum Mechanics

Before we can discuss RQM, we first need to introduce a notation. Let \( \mathbf{u} = a + x + f + v + b \), where \( a \) is a scalar, \( x \) is a vector, \( f \) is a bivector, \( v \) is a pseudo-vector and \( b \) is a pseudo-scalar, be a multivector of the geometric algebra GA(3, 1), and let \( \mathbf{M} \) be its matrix representation. Then, the fundamental constraint of RQM is:

**QG Constraint 1 of 1 Vanishing Relativistic Phase:** Our formulation of QG is based around a vanishing phase spanning the \( \text{Spin}^c(3, 1) \) group. The constraint is:

\[
0 = \text{tr} \frac{1}{2} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}(q)
\]

where \( \mathbf{M} \) is the matrix representation of the multivector \( \mathbf{u} = f + b \) of GA(3, 1). Using the real Majorana representation of the gamma matrices, the representation is as follows:

\[
\mathbf{M} = \begin{bmatrix}
    f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\
    -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\
    -f_{01} - f_{13} & f_{03} - f_{23} & -f_{02} & b - f_{13} \\
    f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02}
\end{bmatrix}
\]

Similarly to the QM case, here the matrix representation engenders a \( \text{Spin}^c(3, 1) \)-phase and the trace will cause it to vanish under measurement.

The Lagrange multiplier equation is as follows:

**Definition 3 (Fundamental Lagrange Multiplier Equation of RQM).**

\[
\mathcal{L}(\rho, \lambda, \zeta) = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left( 1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \zeta \left( -\text{tr} \frac{1}{2} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}(q) \right)
\]

where \( \lambda \) and \( \zeta \) are the Lagrange multipliers.

**Theorem 3.** The least biased probability measure that connects an initial preparation \( p(q) \) to its final measurement \( \rho(q) \), under the constraint of the vanishing relativistic phase, is:

\[
\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp(-\zeta \frac{1}{2} \mathbf{M}(r))} \det \exp(-\zeta \frac{1}{2} \mathbf{M}(q)) p(q)
\]

The proof of this theorem is presented in the results section.
In the results section, we aim to demonstrate that this solution represents a quantum mechanical theory of inertial reference frames, where $\zeta$ is a one-parameter generator of boosts, rotations, and phase transformations. This theory allows for measurements, superpositions, and interference between inertial reference frames, providing the arena in which RQM lives.

1.3 Quantum Gravity

The maximization problem employed for RQM (as described in Section 1.2) will also be found to encompass QG. While RQM was situated within a subset of the solution space, fully leveraging the generality of the solution is necessary for QG. Specifically, the probability measure solving the problem will be found to incorporate a double-copy mechanism applied to the Dirac current, where two Dirac currents are multiplied together to yield a tensor. In this context, a single copy of the Dirac current pertains to RQM, whereas the double-copy leads to QG. The double-copy of the Dirac current facilitates the construction of the metric tensor as an observable from basis vectors, thereby establishing a connection between the probability measure and the geometrical structure of spacetime.

1.4 Dimensional Obstructions

We end the results section with a number of theorems showing that the entropy maximization technique, except for SM (no vanishing phase) and QM (vanishing U(1) phase), yields a solution only in 3+1-dimensional spacetime (vanishing Spin$^c$(3, 1) phase), encountering various obstructions in all other dimension configurations, and we discuss the implications.

2 Results

2.1 Quantum Mechanics

In statistical mechanics, the founding observation is that energy measurements of a thermally equilibrated system tend towards an average value. Comparatively, in QM, the founding observation involves the interplay between the systematic elimination of complex phases in measurement outcomes and the presence of interference effects in repeated measurement outcomes. To represent this observation, we introduce the Vanishing U(1)-Phase Anti-Constraint:

$$0 = \text{tr} \sum_{q \in Q} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}$$

(11)

where $E(q)$ are scalar-valued functions of $Q$. The usage of the matrix generates a U(1) phase, and the trace causes it to vanish under measurements. At first glance, this expression may seem to reduce to a tautology equating zero with zero, suggesting it imposes no restriction on energy measurements.
However, this appearance is deceptive. Unlike a conventional constraint that limits the solution space, this expression serves as a formal device to expand it, allowing for the incorporation of complex phases into the probability measure. The expression’s role in broadening, rather than restricting, the solution space leads to its designation as an "anti-constraint."

In general, usage of anti-constraints expand classical probability measures into larger domains, such as quantum probabilities.

Its significance will become evident upon the completion of the optimization problem. For the moment, this expression can be conceptualized as an ansatz that, when incorporated as an anti-constraint within an entropy-maximization problem, resolves into the axioms of quantum mechanics.

Our next procedural step involves solving the corresponding Lagrange multiplier equation, mirroring the methodology employed in statistical mechanics by E. T. Jaynes. We utilize the relative Shannon entropy because we wish to solve for the least biased probability measure that connects an initial preparation \( p(q) \) to its final measurement \( \rho(q) \). For that, we deploy the following Lagrange multiplier equation:

\[
\mathcal{L} = -\sum_{q \in \mathcal{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left( 1 - \sum_{q \in \mathcal{Q}} \rho(q) \right) + \tau \left( \text{tr} \sum_{q \in \mathcal{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)
\]

(12)

Where \( \lambda \) and \( \tau \) are the Lagrange multipliers.

We solve the maximization problem as follows:

\[
\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - p(q) - \lambda - \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}
\]

(13)

\[
0 = \ln \frac{\rho(q)}{p(q)} + p(q) + \lambda - \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}
\]

(14)

\[\Rightarrow \ln \frac{\rho(q)}{p(q)} = -p(q) - \lambda - \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\]

(15)

\[\Rightarrow \rho(q) = p(q) \exp(-p(q) - \lambda) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)\]

(16)

\[= \frac{1}{Z(\tau)}p(q) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)\]

(17)
The partition function, is obtained as follows:

\[ 1 = \sum_{r \in Q} p(r) \exp(-p(q) - \lambda) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \]  

(18)

\[ \implies (\exp(-p(q) - \lambda))^{-1} = \sum_{r \in Q} p(r) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \]  

(19)

\[ Z(\tau) := \sum_{r \in Q} p(r) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \]  

(20)

Finally, the least biased probability measure that connects an initial preparation \( p(q) \) to its final measurement \( \rho(q) \), under the constraint of the vanishing U(1) phase, is:

\[ \rho(q) = \frac{1}{\sum_{r \in Q} p(r) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \exp \left( -\tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) p(q) } \]  

(21)

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

Upon examination, we find that phase elimination is manifestly evident in the probability measure: since the trace evaluates to zero, the probability measure simplifies to classical probabilities, aligning precisely with the Born rule’s exclusion of complex phases:

\[ \rho(q) = \frac{p(q)}{\sum_{r \in Q} p(r)} \]  

(22)

However, the significance of this phase elimination extends beyond this mere simplicity. As we will soon see, the partition function \( Z \) gains unitary invariance, allowing for the emergence of interference patterns and other quantum characteristics under appropriate basis changes.

We will begin by aligning our results with the conventional quantum mechanical notation. As such, we transform the representation of complex numbers from \( \begin{bmatrix} a \\ b \end{bmatrix} \) to \( a + ib \). For instance, the exponential of a complex matrix is:

\[ \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos(b) - \sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \]  

(23)

Then, we associate the exponential trace to the complex norm using \( \exp \text{tr} M \equiv \det \exp M \):

\[ \exp \text{tr} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r^2 \det \begin{bmatrix} \cos(b) - \sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \]  

(24)

\[ = r^2 (\cos^2(b) + \sin^2(b)) \]  

(25)

\[ = \| r (\cos(b) + i \sin(b)) \| \]  

(26)

\[ = \| r \exp(ib) \| \]  

(27)
Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_BT)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial preparation into:

$$p(q) = \frac{1}{\sum_{r \in Q} p(r) \|\exp(-iE(r)/\hbar)\|} \|\exp(-iE(q)/\hbar)\| p(q)$$

(28)

Unitarily Invariant Partition Function

We are now in a position to explore the solution space.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_{r \in Q} p(r) \|\exp(-iE(r)/\hbar)\| = Z = \langle \psi | \psi \rangle$$

(29)

where

$$\begin{bmatrix}
\psi_1(t) \\
\vdots \\
\psi_n(t)
\end{bmatrix} = \begin{bmatrix}
\exp(-iE(q_1)/\hbar) \\
\vdots \\
\exp(-iE(q_n)/\hbar)
\end{bmatrix} \begin{bmatrix}
\psi_1(0) \\
\vdots \\
\psi_n(0)
\end{bmatrix}$$

(30)

We clarify that $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \langle \psi_i(0)|\psi_i(0) \rangle$.

We also note that $Z$ is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.

- In $Z$, an observable must satisfy:

$$\overline{O} = \sum_{r \in Q} p(r) O(r) \|\exp(-iE(r)/\hbar)\|$$

(31)

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle O\psi | \phi \rangle = \langle \psi | O\phi \rangle$ will equate the above equation, simply because $\langle O \rangle = \langle \psi | O | \psi \rangle$. This demonstrates QM Axiom 2 of 5.

- Upon transforming Equation 30 out of its eigenbasis through unitary operations, we find that the energy, $E(q)$, typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-itH/\hbar)|\psi(0)\rangle$$

(32)
The system’s dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

\[
\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) \\
= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \\
= -i\mathbf{H}/\hbar|\psi(t)\rangle
\]

\[
\implies \mathbf{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle
\]

which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.

- From Equation 30 it follows that the possible microstates \(E(q)\) of the system correspond to specific eigenvalues of \(\mathbf{H}\). An observation can thus be conceptualized as sampling from \(\rho(q,t)\), with the measured state being the occupied microstate \(q\) of \(\Omega\). Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of \(\mathbf{H}\). Measured in the eigenbasis, the probability measure is:

\[
\rho(q,t) = \frac{1}{\langle \psi|\psi \rangle} (\psi(q,t))^\dagger \psi(q,t).
\]

In scenarios where the probability measure \(\rho(q,\tau)\) is expressed in a basis other than its eigenbasis, the probability \(P(\lambda_i)\) of obtaining the eigenvalue \(\lambda_i\) is given as a projection on a eigenstate:

\[
P(\lambda_i) = |\langle \lambda_i|\psi \rangle|^2
\]

Here, \(|\langle \lambda_i|\psi \rangle|^2\) signifies the squared magnitude of the amplitude of the state \(|\psi \rangle\) when projected onto the eigenstate \(|\lambda_i \rangle\). As this argument hold for any observables, this demonstrates QM Axiom 4 of 5.

- Finally, since the probability measure (Equation 28) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the vanishing \(U(1)\) phase constraint (Equation 11) is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. Equation 11 becomes the formulation’s new singular foundation, and Axioms 1, 2, 3, 4, and 5 are now promoted to theorems.

### 2.2 RQM in 2D

In this section, we investigate RQM in 2D. Although all dimensional configurations except 3+1D contain obstructions, which will be discussed later in this section, the 2D case provides a valuable starting point before addressing the
more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier
Equation is:

\[
\mathcal{L}(\rho, \lambda, \theta) = -\sum_{q \in \mathcal{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left( 1 - \sum_{q \in \mathcal{Q}} \rho(q) \right) + \theta \left( -\text{tr} \frac{1}{2} \sum_{q \in \mathcal{Q}} \rho(q) \mathbf{M}(q) \right)
\]

(39)

where \( \lambda \) and \( \theta \) are the Lagrange multipliers, and where \( \mathbf{M}(q) \) is the matrix representation of a multivector \( \mathbf{b}(q) \) of GA(2), where \( \mathbf{b} \) is a pseudo-scalar. In general a multivector \( \mathbf{u} = a + \mathbf{x} + \mathbf{b} \) of GA(2), where \( a \) is a scalar, \( \mathbf{x} \) is a vector and \( \mathbf{b} \) a pseudo-scalar, is represented as follows:

\[
\begin{bmatrix}
a + x & y - b \\
y + b & a - x
\end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y
\]

(40)

The basis elements are defined as:

\[
\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

(41)

If we take \( a \to 0, \mathbf{x} \to 0 \) then \( \mathbf{M} \) reduces as follows:

\[
\mathbf{u} = a + \mathbf{x} + \mathbf{b} |_{a \to 0, \mathbf{x} \to 0} = \mathbf{b} \Rightarrow \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}
\]

(42)

The Lagrange multiplier equation can be solved as follows:

\[
\frac{\partial \mathcal{L}(\rho, \lambda, \theta)}{\partial \rho(q)} = 0 = -\ln \frac{\rho(q)}{p(q)} - p(q) - \lambda - \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}
\]

(43)

\[
0 = \ln \frac{\rho(q)}{p(q)} + p(q) + \lambda + \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}
\]

(44)

\[
\Rightarrow \ln \frac{\rho(q)}{p(q)} = -p(q) - \lambda - \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}
\]

(45)

\[
\Rightarrow \rho(q) = p(q) \exp(-p(q) - \lambda) \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right)
\]

(46)

\[
= \frac{1}{Z(\theta)} p(q) \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right)
\]

(47)

The partition function \( Z(\theta) \), serving as a normalization constant, is deter-
mined as follows:

\[ 1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-p(q) - \lambda) \exp \left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right) \]  

(48)

\[ \implies (\exp(-p(q) - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp \left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right) \]  

(49)

\[ Z(\theta) := \sum_{r \in \mathbb{Q}} p(r) \exp \left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right) \]  

(50)

Consequently, the least biased probability measure that connects an initial preparation \( p(q) \) to a final measurement \( \rho(q) \), under the constraint of the vanishing relativistic phase in 2D is:

\[
\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp \left(-\frac{1}{2} \theta \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right)} \det \exp \left(-\frac{1}{2} \theta \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right) p(q)
\]

(51)

where \( \det \exp M = \exp \operatorname{tr} M \).

In 2D, the Lagrange multiplier \( \theta \) correspond to an angle of rotation, and in 1+1D it would correspond to the rapidity \( \zeta \):

\[
2D: \quad \exp \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \theta - \sin \theta \\ \sin \theta \cos \theta \end{bmatrix} \quad \theta \text{ is the angle of rotation}
\]

(52)

\[
1 + 1D: \quad \exp \zeta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cosh \zeta \sinh \zeta \\ \sinh \zeta \cosh \zeta \end{bmatrix} \quad \zeta \text{ is the rapidity}
\]

(53)

The 2D solution may appear equivalent to the QM case because they are related by an isomorphism \( \operatorname{Spin}(2) \cong \operatorname{SO}(2) \cong \operatorname{U}(1) \) and under the replacement \( \theta \to \tau \). However, an isomorphism does not mean identical, and in \( \operatorname{Spin}(2) \) we gain extra structures related to a relativistic description, which are not available in the QM case.

To investigate the solution in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

**Definition 4** (Multivector conjugate (a.k.a Clifford conjugate)). Let \( u = a + x + b \) be a multi-vector of the geometric algebra over the reals in two dimensions \( \operatorname{GA}(2) \). The multivector conjugate is defined as:

\[ u^\dagger = a - x - b \]

(54)

The determinant of the matrix representation of a multivector can be expressed as a self-product:
Theorem 4 (Determinant as a Multivector Self-Product).

\[ \mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M} \quad (55) \]

Proof. Let \( \mathbf{u} = a + x \sigma_x + y \sigma_y + b \sigma_x \wedge \sigma_y \), and let \( \mathbf{M} \) be its matrix representation \([a+x \quad y-b] \quad [y+b \quad a-x] \). Then:

1: \( \mathbf{u}^\dagger \mathbf{u} \)

\[
= (a + x \sigma_x + y \sigma_y + b \sigma_x \wedge \sigma_y)^\dagger (a + x \sigma_x + y \sigma_y + b \sigma_x \wedge \sigma_y) \\
= (a - x \sigma_x - y \sigma_y - b \sigma_x \wedge \sigma_y)(a + x \sigma_x + y \sigma_y + b \sigma_x \wedge \sigma_y) \\
= a^2 - x^2 - y^2 + b^2 
\]

2: \( \det \mathbf{M} \)

\[
= \det \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \\
= (a + x)(a - x) - (y - b)(y + b) \\
= a^2 - x^2 - y^2 + b^2 
\]

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 5 (Multivector Conjugate Transpose). Let \( |V\rangle \in (\text{GA}(2))^n \):

\[
|V\rangle = \begin{bmatrix} a_1 + x_1 + b_1 \\ \vdots \\ a_n + x_n + b_n \end{bmatrix} 
\]  

The multivector conjugate transpose of \( |V\rangle \) is defined as first taking the transpose and then the element-wise multivector conjugate:

\[
\langle V | = [a_1 - x_1 - b_1 \ldots a_n - x_n - b_n] 
\]  

Definition 6 (Bilinear Form). Let \( |V\rangle \) and \( |W\rangle \) be two vectors valued in \( \text{GA}(2) \). We introduce the following bilinear form:

\[
\langle V | W \rangle = (a_1 - x_1 - b_1)(a_1 + x_1 + b_1) + \ldots (a_n - x_n - b_n)(a_n + x_n + b_n) 
\]

Theorem 5 (Inner Product). Restricted to the even sub-algebra of \( \text{GA}(2) \), the bilinear form is an inner product.

Proof.

\[
\langle V | W \rangle_{x \to 0} = (a_1 - b_1)(a_1 + b_1) + \ldots (a_n - b_n)(a_n + b_n) 
\]

This is isomorphic to the inner product of a complex Hilbert space, with the identification \( i \cong \sigma_x \wedge \sigma_y \). \( \Box \)
Definition 7 (Spin(2)-valued Wavefunction).

\[ |\psi\rangle = \left[ e^{\frac{i}{2}(a_1+b_1)} \cdots e^{\frac{i}{2}(a_n+b_n)} \right] = \left[ \sqrt{\rho_1} R_1 \cdots \sqrt{\rho_2} R_2 \right] \tag{68} \]

where \( \sqrt{\rho_i} = e^{\frac{i}{2}a_i} \) representing the square root of the probability and \( R_i = e^{\frac{i}{2}b_i} \) representing a rotor in 2D (or boost in 1+1D).

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

**Theorem 6 (Partition Function).** \( Z = \langle \psi|\psi \rangle \)

**Proof.**

\[ \langle \psi|\psi \rangle = \sum_{q \in Q} \psi(q)^\dagger \psi(q) = \sum_{q \in Q} \rho(q) R(q)^\dagger R(q) = \sum_{q \in Q} \rho(q) = Z \tag{69} \]

\( \square \)

Thus, the Spin(2)-valued wavefunction \( |\psi\rangle \) is a linear object whose inner product reduces to the partition function.

**Definition 8 (Spin(2)-valued Evolution Operator).**

\[ T = \begin{bmatrix} e^{-\frac{i}{2} \theta b_1} & & \\ & \ddots & \\ & & e^{-\frac{i}{2} \theta b_n} \end{bmatrix} \tag{70} \]

**Theorem 7.** The partition function is invariant with respect to the Spin(2)-valued evolution operator.

**Proof.**

\[ \langle T\psi|T\psi \rangle = \sum_{q \in Q} \det(T(q)\psi(q)) = \sum_{q \in Q} \det T(q) \det \psi(q) = \sum_{q \in Q} \det \psi(q) = \langle \psi|\psi \rangle \tag{71} \]

where \( \det T(q) = 1 \), because \( e^{-\frac{i}{2} \theta b(q)} \) is traceless. \( \square \)

We note that since the even sub-algebra of GA(2) is closed under addition and multiplication, and the bilinear form constitutes an inner product, it follows that it can be employed to construct a Hilbert space, in this case a Spin(2)-valued Hilbert space. The primary difference between a wavefunction living in a complex Hilbert space and one living in a Spin(2) Hilbert space relates to the subject matter of the theory. In the present case, the subject matter is a quantum theory of inertial reference frames in 2D.
The dynamics of reference frame transformations follow from the Schrödinger equation, which is obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier $\theta$. Each element of the wavefunction represents an inertial reference frame, whose transformation is generated by the $\theta$ angle (for instance, the change of angle experienced by an inertial observer).

**Definition 9** (Spin(2)-valued Schrödinger Equation).

\[
\frac{d}{d\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}b_1 \\ \vdots \\ -\frac{1}{2}b_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \tag{72}
\]

Here, $\theta$ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the evolution group of the wavefunction.

**Definition 10** (Reference Frame Measurement). The QM Axiom 5 of 5, regarding the measurement postulates, is derived as a theorem in the RQM case as well (for the same reason as it is in the QM case). This allows us to measure the wavefunction $|\psi\rangle$ into one of its states $q$ according to probability $\rho(q)$. Here the post-measurement state $q$ corresponds to picking a specific inertial reference frame $q$ from $Q$.

We note that, as a linear system, linear combinations of the wavefunction (such as $\psi(q) = \lambda_1\psi_1(q) + \lambda_2\psi_2(q)$) will also be solutions. This can introduce interference patterns between inertial reference frames:

**Theorem 8** (Reference Frame Superpositions and Interference).

*Proof.* Let $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\rho_1}R_1 \\ \sqrt{\rho_2}R_2 \end{bmatrix}$, then:

\[
T|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\rho_1}R_1 \\ \sqrt{\rho_2}R_2 \end{bmatrix} = \frac{1}{2} \left( \sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2 \right) |0\rangle + \frac{1}{2} \left( \sqrt{\rho_1}R_1 - \sqrt{\rho_2}R_2 \right) |1\rangle \tag{73}
\]

Then the probability can be computed as follows:

\[
|\langle 0|\psi\rangle|^2 = \frac{1}{2}(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2)\left(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2\right) = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \frac{1}{2}\sqrt{\rho_1\rho_2}(R_1^\dagger R_2 + R_2^\dagger R_1) \tag{76}
\]

\[
= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \frac{1}{2}\sqrt{\rho_1\rho_2}\cos(\theta b_1 - \theta b_2) \tag{77}
\]

Spin(2)-valued Interference
Since Spin(2) \cong U(1), then Spin(2)-valued interference is isomorphic to complex interference.

**Definition 11** (David Hestenes’ Formulation). In 3+1D, the David Hestenes’ formulation \cite{hestenes} of the wavefunction is \( \psi = \sqrt{\rho}Re^{ib/2} \), where \( R = e^{f/2} \) is a Lorentz boost or rotation and where \( e^{ib/2} \) is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to \( \psi = \sqrt{\rho}R \), which is identical to what we recovered.

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

**Definition 12** (Dirac Current). Given the basis \( \sigma_x \) and \( \sigma_y \), the Dirac current for the 2D theory is defined as:

\[
J_1 \equiv \psi^\dagger \sigma_x \psi = \rho R^\dagger \sigma_x R = \rho \tilde{\sigma}_x \tag{79}
\]

\[
J_2 \equiv \psi^\dagger \sigma_y \psi = \rho R^\dagger \sigma_y R = \rho \tilde{\sigma}_y \tag{80}
\]

where \( \tilde{\sigma}_x \) and \( \tilde{\sigma}_y \) are a SO(2) rotated basis vectors.

### 2.2.1 Obstructions

As stated, all dimensional configurations except 3+1D contain obstructions. Specifically, in 1+1D and 2D, we identify two obstructions:

1. **In 1+1D:** The 1+1D theory results in a split-complex quantum theory due to the bilinear form \((a - be_0 \wedge e_1)(a + be_0 \wedge e_1)\), which yields negative probabilities: \( a^2 - b^2 \in \mathbb{R} \) for certain wavefunction states, in contrast to the non-negative probabilities \( a^2 + b^2 \in \mathbb{R}_{\geq 0} \) obtained in the Euclidean 2D case. (This is why we had to use 2D instead of 1+1D in this two-dimensional introduction...)

2. **In 1+1D and in 2D:** The basis vectors \( \sigma_x \) and \( \sigma_y \) in 2D, and \( e_0 \) and \( e_1 \) in 1+1D) are not self-adjoint. Although used in the context defining the Dirac current, their non-self-adjointness prevents the construction of the pseudo-Riemannian inner product as a quantum observable. The benefits of having the basis vectors self-adjoint will become obvious in the 3+1D case, where we will be able to construct the metric tensor from inner product measurements. Specifically, in 2D:

\[
(e_\mu u)^\dagger u \neq u^\dagger e_\mu u \tag{81}
\]

because \( (e_\mu u)^\dagger u = u^\dagger e_\mu^\dagger u = u^\dagger (-e_\mu)u \).

In the following section, we will explore the obstruction-free 3+1D case.
2.3 RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L}(\rho, \lambda, \tau) = -\sum_{q \in Q} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left( 1 - \sum_{q \in Q} \rho(q) \right) + \zeta \left( -\text{tr} \frac{1}{2} \sum_{q \in Q} \rho(q) M(q) \right)$$

(82)

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho(q) = \frac{1}{\sum_{r \in Q} p(r) \det \exp(-\zeta \frac{1}{2} M(r))} \det \exp(-\zeta \frac{1}{2} M(q)) \frac{p(q)}{\text{Spin}^c(3,1) \text{ Born Rule}}$$

(83)

where $\zeta$ is a "twisted-phase" rapidity. (If the invariance group was Spin(3,1) instead of Spin$^c(3,1)$, obtainable by posing $b \to 0$, then it would simply be the rapidity).

2.3.1 Preliminaries

Our initial goal will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra GA(3,1).

**Definition 13 (Multivector).** Let $u$ be a multivector of GA(3,1). Its general form is:

$$u = a + x + f + v + b$$

(84)

where $a, x, f$ are scalars, vectors, bivectors, pseudo-vectors and pseudo-scalars.

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3$$

(85)

$$+ f_0 \gamma_0 \wedge \gamma_1 + f_0 \gamma_0 \wedge \gamma_2 + f_0 \gamma_0 \wedge \gamma_3 + f_1 \gamma_1 \wedge \gamma_2 + f_1 \gamma_1 \wedge \gamma_3 + f_2 \gamma_2 \wedge \gamma_3$$

(86)

$$+ p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2$$

(87)

$$+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$$

(88)

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are the basis vectors in the real Majorana representation.

A more compact notation for $u$ is:

$$u = a + x + f + v + b$$

(89)
This general multivector can be represented by a $4 \times 4$ real matrix using the real Majorana representation:

**Definition 14** (Matrix Representation of $u$).

$$
M = \begin{bmatrix}
  a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\
  -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\
  -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\
  f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \\
\end{bmatrix}
$$

(90)

To manipulate and analyze multivectors in GA(3,1), we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

**Definition 15** (Multivector Conjugate (in 4D)).

$$
u^\dagger = a - x - f + v + b
$$

(91)

**Definition 16** (3,4 Blade Conjugate). The 3,4 blade conjugate of $u$ is

$$
|u|_{3,4} = a + x + f - v - b
$$

(92)

The results of Lundholm[8], demonstrates that the multivector norms in the following definition, are the unique forms which carries the properties of the determinants such as $N(uv) = N(u)N(v)$ to the domain of multivectors:

**Definition 17.** The self-products associated with low-dimensional geometric algebras are:

- $\text{GA}(0,1): \quad \varphi^\dagger \varphi$
- $\text{GA}(2,0): \quad \varphi^\dagger \varphi$
- $\text{GA}(3,0): \quad [\varphi^\dagger \varphi]_3 \varphi^\dagger \varphi$
- $\text{GA}(3,1): \quad [\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi$
- $\text{GA}(4,1): \quad ([\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi)^{(}[\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi)$

(93) - (97)

We can now express the determinant of the matrix representation of a multivector via the self-product $[\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi$. Again, this choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within GA(3,1):

**Theorem 9** (Determinant as a Multivector Self-Product).

$$
|u^\dagger u|_{3,4} u^\dagger u = \det M
$$

(98)

*Proof.* Please find a computer assisted proof of this equality in Annex C.  

□
Definition 18 (GA(3, 1)-valued Vector).

\[ |V\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a_1 + x_1 + f_1 + v_1 + b_1 \\ \vdots \\ a_n + x_n + f_n + v_n + b_n \end{bmatrix} \]  

(99)

These constructions allow us to express the partition function in terms of the multivector self-product:

Definition 19 (Double-Copy Product). Instead of an inner product, we obtain what we call a double-copy product:

\[ \langle V|V|V \rangle = \sum_{q \in Q} \left( \psi(q)^\dagger \psi(q) \right)_{3,4} \left( \psi(q)^\dagger \psi(q) \right) \]  

(100)

\[ = \begin{bmatrix} u_1^\dagger & \ldots & u_n^\dagger \end{bmatrix} \begin{bmatrix} u_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & u_n \end{bmatrix}_{3,4} \begin{bmatrix} u_1^\dagger \\ \vdots \\ u_n^\dagger \end{bmatrix} \] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \]  

(101)

Theorem 10 (Partition Function). \( Z = \langle V|V|V \rangle \)

Proof.

\[ \langle V|V|V \rangle = \sum_{q \in Q} \left( \psi(q)^\dagger \psi(q) \right)_{3,4} \left( \psi(q)^\dagger \psi(q) \right) \]  

(102)

\[ = \begin{bmatrix} u_1^\dagger & \ldots & u_n^\dagger \end{bmatrix} \begin{bmatrix} u_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & u_n \end{bmatrix}_{3,4} \begin{bmatrix} u_1^\dagger \\ \vdots \\ u_n^\dagger \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \]  

(103)

\[ = \begin{bmatrix} [u_1^\dagger u_1] & \ldots & [u_n^\dagger u_n] \end{bmatrix}_{3,4} \begin{bmatrix} u_1^\dagger \\ \vdots \\ u_n^\dagger \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \]  

(104)

\[ = [u_1^\dagger u_1]_{3,4} [u_1^\dagger u_1] + \cdots + [u_n^\dagger u_n]_{3,4} [u_n^\dagger u_n] \]  

(105)

\[ = \sum_{i=1}^{n} \det M_{u_i} \]  

(106)

\[ = Z \]  

(107)

Desirable properties for the double-copy product are introduced by reducing multivectors to its subgroups. First, non-negativity:

Theorem 11 (Non-negativity). The double-copy product, applied to the even sub-algebra of GA(3, 1) is always non-negative.
Proof. Let $|V\rangle = \begin{bmatrix} a_1 + f_1 + b_1 \\ \vdots \\ a_n + f_n + b_n \end{bmatrix}$. Then,

$$
\langle V | V | V | V \rangle (108)
= \begin{bmatrix} (a_1 + f_1 + b_1)(a_1 + f_1 + b_1) \\ \vdots \\ (a_1 + f_1 + b_1)(a_1 + f_1 + b_1) \end{bmatrix}
= \begin{bmatrix} (a_1 - f_1 + b_1)(a_1 + f_1 + b_1) \\ \vdots \\ (a_1 - f_1 + b_1)(a_1 + f_1 + b_1) \end{bmatrix}
= \begin{bmatrix} (a_1^2 + a_1f_1 + a_1b_1 - f_1a_1 - f_1^2 - f_1b_1 + b_1a_1 + b_1f_1 + b_1^2) \\ \vdots \\ (a_1^2 - f_1^2 + b_1^2) \end{bmatrix}
$$

(109)

(110)

(111)

(112)

We note 1) $b^2 = (bI)^2 = -b^2$ and 2) $f^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$
= \begin{bmatrix} (a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) \end{bmatrix}
$$

(113)

We note that the terms are now complex numbers, which we rewrite as $Re(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $Im(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$
= \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}
$$

(114)

(115)

(116)

which is always non-negative.

Then, positive-definiteness of the double-copy product is obtained by creating an equivalence class between the zero vector and any non-zero vector of length zero, and taking the zero vector as the representative of the class. To realize the equivalence class, we define the Spin$^c(3, 1)$-valued wavefunction, which is valued in the even sub-algebra of GA$(3, 1)$, as follows:
Definition 20 (Spin$^c(3, 1)$-valued Wavefunction).

\[
|\psi\rangle = \begin{bmatrix}
e^{\frac{i}{2}(a_1+f_1+b_1)} \\
\vdots \\
e^{\frac{i}{2}(a_n+f_n+b_n)}
\end{bmatrix} = \begin{bmatrix}\sqrt{\rho_1}B_1 \\
\vdots \\
\sqrt{\rho_n}B_n\end{bmatrix}
\]

(117)

where $R_i = e^{\frac{i}{2}f_i}$ is a rotor, $B_i = e^{\frac{i}{2}b_i}$ is a phase, and where $\sqrt{\rho_i} = e^{\frac{i}{2}a_i} \geq 0$.

Any even multivectors of GA(3,1) admits a unique exponential representation, except when $\rho_i = 0$ in which it is surjective. Consequently, in this representation the double-copy product yields 0 only for the zero vector, rendering the double-copy product positive-definite.

Now, let us turn our attention to the evolution operator, which leaves the partition function invariant:

Definition 21 (Spin$^c(3, 1)$ Evolution Operator).

\[
T = \begin{bmatrix}e^{-\frac{i}{2}\zeta f_1 + b_1} \\
\vdots \\
e^{-\frac{i}{2}\zeta f_n + b_n}\end{bmatrix}
\]

(118)

In turn, this leads to a Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier $\zeta$:

Definition 22 (Spin$^c(3, 1)$-valued Schrödinger equation).

\[
\frac{d}{d\zeta} \begin{bmatrix}\psi_1(\zeta) \\
\vdots \\
\psi_n(\zeta)\end{bmatrix} = \begin{bmatrix}-\frac{i}{2}(f_1 + b_1) \\
\vdots \\
-\frac{i}{2}(f_n + b_n)\end{bmatrix} \begin{bmatrix}\psi_1(\zeta) \\
\vdots \\
\psi_n(\zeta)\end{bmatrix}
\]

(119)

In this case $\zeta$ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin$^c(3, 1)$, locally across the states of the vector space. This is an extremely general equation that captures all transformations that can be done consistently with the evolution group of the wavefunction.

Finally, we note that since the probability measure is invariant with respect to the evolution operator, then it follows that the dynamics of the theory are invariant with respect to action by the Spin$^c(3, 1)$ group.

2.3.2 RQM in 3+1D

Definition 23 (David Hestenes’ Formulation). The Spin$^c(3, 1)$-valued wavefunction we have recovered is identical to David Hestenes’ formulation of the wavefunction within GA(3,1).

\[
\psi = e^\frac{i}{2}(a + f + b) = \sqrt{\rho}Re^{-ib/2}
\]

(120)

where $e^{\frac{i}{2}a} = \sqrt{\rho}$, $e^{\frac{i}{2}f} = R$ and $e^{\frac{i}{2}b} = e^{-ib/2}$. 

20
Before we continue the RQM investigation, let us note that the double-copy product contains two copies of a bilinear form $\psi^\dagger \psi$:

\[
\begin{array}{c|c}
\psi^\dagger & \psi \\
\hline
\text{copy 1} & \text{copy 2}
\end{array}
\]  

(121)

In this section regarding RQM in 3+1D, we will investigate the properties of each copy individually, leaving the properties specific to the double-copy for the section on quantum gravity.

Taking a single copy, the Dirac current is obtained directly from the gamma matrices, as follows:

**Definition 24** (Dirac Current). The definition of the Dirac current is the same as Hestenes:

\[
J \equiv \psi^\dagger \gamma_\mu \psi = \rho R^\dagger B^1 \gamma_\mu BR = \rho R^\dagger \gamma_\mu B^{-1} BR = \rho R^\dagger \gamma_\mu \overset{\text{SO}(3,1)}{=} \hat{\gamma}_\mu
\]  

(122)

where $\hat{\gamma}_\mu$ is a SO(3,1) rotated basis vector.

We will now demonstrate that the copied bilinear form is invariant with respect to the U(1), SU(2), and SU(3) symmetries and the Spin(3,1) and unitary $U^\dagger U = I$ symmetries which play a fundamental role in the standard model of particle physics.

**Theorem 12** (U(1) Invariance). Let $e^{\frac{1}{2}b}$ be a general element of U(1). Then, the equality

\[
|\psi^\dagger \gamma_0 \psi|_{3,4} \psi^\dagger \gamma_0 \psi = |(e^{\frac{1}{2}b} \psi)^\dagger \gamma_0 e^{\frac{1}{2}b} \psi|_{3,4} (e^{\frac{1}{2}b} \psi)^\dagger \gamma_0 e^{\frac{1}{2}b} \psi
\]  

(123)

is satisfied, yielding a U(1) symmetry for each copied bilinear form.

**Proof.** Equation 123 is invariant if this expression is satisfied:

\[
e^{\frac{1}{2}b} \gamma_0 e^{\frac{1}{2}b} = \gamma_0
\]  

(124)

This is always satisfied simply because $e^{\frac{1}{2}b} \gamma_0 e^{\frac{1}{2}b} = \gamma_0 e^{-\frac{1}{2}b} e^{\frac{1}{2}b} = \gamma_0$

**Theorem 13** (SU(2) Invariance). Let $e^{\frac{1}{2}f}$ be a general element of Spin(3,1). Then, the equality:

\[
|\psi^\dagger \gamma_0 \psi|_{3,4} \psi^\dagger \gamma_0 \psi = |(e^{\frac{1}{2}f} \psi)^\dagger \gamma_0 e^{\frac{1}{2}f} \psi|_{3,4} (e^{\frac{1}{2}f} \psi)^\dagger \gamma_0 e^{\frac{1}{2}f} \psi
\]  

(125)

is satisfied for if $f = \theta_1 \gamma_0 \gamma_1 + \theta_2 \gamma_0 \gamma_2 + \theta_3 \gamma_0 \gamma_3$ (which generates SU(2)), yielding a SU(2) symmetry for each copied bilinear form.
Proof. Equation 125 is invariant if this expression is satisfied[9]:

\[ e^{-\frac{i}{2}\gamma_0}e^{\frac{i}{2}\gamma} = \gamma_0 \]  

(126)

We now note that moving the left-most term to the right of the gamma matrix yields:

\[ e^{-\theta_1\gamma_0\gamma_1 - \theta_2\gamma_0\gamma_2 - \theta_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2\gamma_3\gamma_3 - B_3\gamma_1\gamma_2}e^{\frac{i}{2}\gamma} = \gamma_0 e^{-\theta_1\gamma_0\gamma_1 - \theta_2\gamma_0\gamma_2 - \theta_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_3\gamma_3 + B_3\gamma_1\gamma_2}e^{\frac{i}{2}\gamma} \]  

(127)

(128)

Therefore, the product \( e^{-\frac{i}{2}\gamma_0}e^{\frac{i}{2}\gamma} \) reduces to \( \gamma_0 \) if and only if \( B_1 = B_2 = B_3 = 0 \), leaving \( \gamma = \theta_1\gamma_1 + \theta_2\gamma_2 + \theta_3\gamma_3 \):

Finally, we note that \( e^{\theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3} \) generates \( SU(2) \). □

**Theorem 14 (SU(3) invariance).** Let \( \gamma = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_3\gamma_3 + B_3\gamma_1\gamma_2 \) be a general bivector. Then, the equality:

\[ [\psi^*\gamma_0\psi]_{1,3,4}\gamma_0\psi = \frac{1}{4}\left( (\bar{\psi}\gamma_0\psi)_{1,3,4}(\bar{\psi}\gamma_0\psi)_{1,3,4} \right) \]  

(129)

is satisfied if \( (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \), where \( E_1\gamma_0\gamma_1 + B_1\gamma_0\gamma_1 I \equiv E_1 + iB_1 \), \( E_2\gamma_0\gamma_2 + B_2\gamma_0\gamma_2 I \equiv E_2 + iB_2 \) and \( E_3\gamma_0\gamma_3 + B_3\gamma_0\gamma_3 I \equiv E_3 + iB_3 \) which are the defining conditions for the \( SU(3) \) symmetry group. This yields a \( SU(3) \) symmetry for each copy of the bilinear form.

**Proof.** Equation 129 is invariant if this expression is satisfied[9, 10]:

\[ -\gamma_0\gamma = \gamma_0 \]  

(130)

which we can rewrite as follows:

\[ -(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_3\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\gamma_0 \]  

(131)

The first three terms anticommute with \( \gamma_0 \), while the last three commute with \( \gamma_0 \):

\[ \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2\gamma_2\gamma_3 - B_3\gamma_2\gamma_3)\gamma_0\gamma_0 \]  

(132)

This can be written as:

\[ \gamma_0(E - B)(E + B) \]  

(133)

\[ = \gamma_0(E^2 + EB - BE - B^2) \]  

(134)

where \( E = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 \) and \( B = B_1\gamma_2\gamma_3 + B_2\gamma_3\gamma_3 + B_3\gamma_1\gamma_2 \).

Thus, for \( -\gamma_0\gamma = \gamma_0 \), we require: 1) \( E^2 - B^2 = 1 \) and 2) \( EB = BE \). The second requirement means that \( E \) and \( B \) must commute (and thus be isomorphic to three complex numbers), and the first implies:

\[ E^2 - B^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \]  

(135)

which are the defining conditions for the \( SU(3) \) symmetry group. □
Theorem 15 (Spin(3,1) invariance). Let $e^{\frac{i}{2}f}$ be a general element of Spin(3,1). Then, the equality:

$$[\psi^\dagger \psi]_{3,4} \psi^\dagger \psi = \underbrace{(e^{\frac{i}{2}f\psi})^\dagger e^{\frac{i}{2}f\psi}}_{\text{copy 1}} \underbrace{(e^{\frac{i}{2}f\psi})^\dagger e^{\frac{i}{2}f\psi}}_{\text{copy 2}}$$ (136)

is always satisfied, yielding a Spin(3,1) symmetry for each copied bilinear form.

Proof. Equation 136 is invariant if this expression is satisfied

$$e^{-\frac{i}{2}f} e^{\frac{i}{2}f} = 1$$ (137)

which is always the case. □

Theorem 16 (Unitary invariance). Let $T$ and $U$ be $n \times n$ unitary matrices. Then unitary invariance

$$\langle \psi | \gamma_\mu \psi | \gamma_\nu \psi \rangle = \langle T \psi | \gamma_\mu T \psi | U \psi | \gamma_\mu U \psi \rangle \implies U^\dagger U = I \text{ and } T^\dagger T = I$$ (138)

is individually satisfied for each copied bilinear form.

Proof. Equation 138 is satisfied if $U^\dagger \gamma_\mu U = \gamma_\mu$, and if $T^\dagger \gamma_\mu T = I$. Without loss of generality, let us do the case for $U$. Since $U$ is valued in complex numbers, then $U^\dagger = U^T$, which reduces the condition to:

$$U^T \gamma_\mu U = \gamma_\mu$$ (139)

Then, since $\gamma_\mu \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_\mu$, it follows that:

$$\gamma_\mu U^\dagger U = \gamma_\mu$$ (140)

which is satisfied when $U^\dagger U = I$. □

These invariances can be promoted to local symmetries using the typical gauge symmetry construction techniques. In standard QM, the Born rule naturally guides us towards a U(1)-valued gauge theory, because of this symmetry: $(e^{-i\theta(x)}\psi(x)) e^{-i\theta(x)} \psi(x) = \psi(x) \psi(x)$. However, the SU(3) and SU(2) symmetries do not emerge from the probability measure in the same manner and must instead be introduced by hand justified by experimental considerations. So why these symmetries and not others? In contrast, within the double-copy product framework, all three symmetry groups—U(1), SU(2), and SU(3)—and also the Spin(3,1) symmetry follow naturally from the invariance of the probability measure, in the same manner that U(1) symmetry follows from the Born rule.
2.4 A Double-Copy Product for Quantum Gravity

In the previous section, we developed a quantum theory of reference frames valued in Spin$_c$(3,1), in which RQM lives. Our goal in this section is to extend the methodology to arbitrary basis vectors, in which the metric tensor lives as an observable. To formulate the theory, we will exploit the features of the double-copy product, which will allow us to formulate the pseudo-Riemannian inner product as an observable from which the metric tensor can be constructed as a double-copy product. Our formulation is reminiscent of the BCJ[11] double copy gauge theory of perturbatively expanded quantum gravity, but our double copy is formulated at the level of the Dirac current.

2.4.1 Initial Investigation

We recall the definition of the metric tensor in terms of basis vectors of geometric algebra, as follows:

\[
g_{\mu\nu} = \frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu)
\]

(141)

Then, we note that the double-copy product acts on a pair of basis element \(e_\mu\) and \(e_\nu\), as follows:

\[
\frac{1}{2} \left( |\psi^\dagger e_\mu \psi\rangle_{3,4} \psi^\dagger e_\nu \psi + |\psi^\dagger e_\nu \psi\rangle_{3,4} \psi^\dagger e_\mu \psi \right)
\]

(142)

\[
= \frac{1}{2} \left( \hat{R} e^{i\beta/2} e^{-i\beta/2} e^{i\beta/2} e^{-i\beta/2} e_\mu \hat{R} e^{i\beta/2} e^{-i\beta/2} e_\nu \hat{R} + \hat{R} e^{i\beta/2} e^{-i\beta/2} e^{i\beta/2} e^{-i\beta/2} e_\nu \hat{R} e^{i\beta/2} e^{-i\beta/2} e_\mu \hat{R} \right)
\]

(143)

\[
= \rho^2 \left( \hat{R} e_\mu \hat{R} e_\nu \hat{R} + \hat{R} e_\nu \hat{R} e_\mu \hat{R} \right)
\]

(144)

\[
= \rho^2 \frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu)
\]

(145)

where \(\hat{e}_\mu\) and \(\hat{e}_\nu\) are SO(3,1) rotated basis vectors.

As one can swap \(e_\mu\) and \(e_\nu\) and obtain the same metric tensor, the double-copy product guarantees that \(g_{\mu\nu}\) is symmetric. Finally, since \(e_\mu^\dagger = -e_\mu\), we get:

\[
(\psi^\dagger e_\mu \psi)_{3,4} (e_\nu \psi)^\dagger \psi
\]

(146)

\[
= |\psi^\dagger (-1) e_\mu^\dagger \psi\rangle_{3,4} \psi^\dagger (-1) e_\nu^\dagger \psi
\]

(147)

\[
= |\psi^\dagger e_\mu \psi\rangle_{3,4} \psi^\dagger e_\nu \psi
\]

(148)

then \(e_\mu\) and \(e_\nu\) are self-adjoint within the double-copy product, entailing the interpretation of \(g_{\mu\nu}\) as an observable.
In the double-copy product, the metric tensor is essentially a double copy of Dirac currents (we recall that the Dirac current is defined as $\rho_{\mu} = \psi^\dagger \gamma_\mu \psi$). As such, it encodes the probabilistic structure of a general relativistic quantum theory in the form of a metric tensor analogously to how the Dirac current encodes the probabilistic structure of a special relativistic quantum theory in the form of a vector.

Let us now investigate the dynamics. We recall that the evolution operator (Definition 21) is:

$$T = \begin{bmatrix}
  e^{-\frac{i}{\hbar}(f_1+b_1)} \\
  \vdots \\
  e^{-\frac{i}{\hbar}(f_n+b_n)}
\end{bmatrix}$$

(149)

Acting on the wavefunction, the effect of this operator cascades down to the basis vectors via the double-copy form:

$$\left[ \psi^d T^4 \gamma_0 T^2 \psi \right]_{\text{copy 1}} \left[ \psi^d T^4 \gamma_1 T^2 \psi \right]_{\text{copy 2}}$$

(150)

which realizes an SO(3,1) transformation of the metric tensor via action of the exponential of a bivector, and a double-copy unitary invariant transformation via action of the exponential of a pseudo-scalar. For instance, a single copy yields:

$$e^{\frac{i}{\hbar} \gamma_0} e^{-\frac{i}{\hbar} \gamma_1} e^{-\frac{i}{\hbar} \gamma_2} e^{-\frac{i}{\hbar} \gamma_3} = e^{\frac{i}{\hbar} \gamma_0} e^{\frac{i}{\hbar} \gamma_1} e^{\frac{i}{\hbar} \gamma_2} e^{\frac{i}{\hbar} \gamma_3} = e^{\frac{i}{\hbar} \gamma_0} e^{-\frac{i}{\hbar} \gamma_1} e^{-\frac{i}{\hbar} \gamma_2} e^{-\frac{i}{\hbar} \gamma_3}$$

(151)

To summarize this initial investigation, we have identified a situation where the wavefunction measures the pseudo-Riemannian inner product by acting on the basis vectors, where the evolution operator, governed by the Schrödinger equation, dynamically realizes SO(3,1) transformations on said metric tensor, and where the amplitudes associated to possible metric tensors are given by a double-copy of unitary quantum theories operating on basis vectors.

Let us now develop these initial observations more rigorously.

### 2.4.2 Starting Point for a Quantum Theory of Gravity

Since the quantum theory operates on basis vectors, we will utilize the Einstein-Hilbert action expressed in terms of the basis vectors:

$$S[e_\mu, e_\nu, \omega] = \frac{c^4}{16\pi G} \int d^4x R(\omega) \left\{ -\det \left( \frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu) \right) \right\}$$

(152)

where $\omega$ is the connection. Similarly to the Palatini action, we can intuit that varying this action yields the EFE when $\omega$ is the Levi-Civita connection, and a modified EFE that can interact with fermions when $\omega$ is the Spin connection.
2.4.3 Linearized Gravity

One path of exploration is to investigate the linearized action. To that end, we can express the basis vector as a perturbation of a flat spacetime metric:

\[ e_\mu = \gamma_\mu + h_\mu \implies g_{\mu\nu} = \frac{1}{2}((\gamma_\mu + h_\mu)(\gamma_\nu + h_\nu) + (\gamma_\nu + h_\nu)(\gamma_\mu + h_\mu)) \]  

(153)

\[ = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\mu h_\nu + h_\mu\gamma_\nu + h_\mu h_\nu + \gamma_\nu h_\mu + \gamma_\mu h_\mu + h_\nu\gamma_\mu + h_\nu h_\mu) \]  

(154)

\[ = \frac{1}{2}(\gamma_\mu\gamma_\nu + \nu_\gamma \gamma_\mu) + \frac{1}{2}(\gamma_\mu h_\nu + h_\mu\gamma_\nu + h_\nu h_\mu + \gamma_\nu h_\mu) + \frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu) \]  

(155)

\[ = \frac{1}{2}(\gamma_\mu\gamma_\nu + \nu_\gamma \gamma_\mu) + \frac{1}{2}(\gamma_\mu h_\nu + h_\mu\gamma_\nu - h_\mu h_\nu - \gamma_\nu h_\mu) + \frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu) \]  

(156)

\[ = \frac{1}{2}(\gamma_\mu\gamma_\nu + \nu_\gamma \gamma_\mu) + \frac{1}{2}(h_\mu h_\nu + h_\nu h_\mu) \]  

(157)

It is well known that working in de Donder gauge \( \partial^\alpha h_{\alpha\mu} - \frac{1}{2}\partial_\mu h = 0 \), where \( h = \eta^{\mu\nu}h_{\mu\nu} \), the Einstein-Hilbert action reduces to its linearized form as follows:

\[ S_{EH}^{(1)}[h_{\mu\nu}] = \int d^4x \left( \frac{1}{2}\partial_\mu h_{\sigma\nu}\partial^\nu h^{\rho\sigma} - \frac{1}{4}\partial_\mu h\partial^\mu h \right) \]  

(158)

Furthermore, varying this action with respect to \( h_{\mu\nu} \) and applying the transverse-traceless gauge, the wave equation follows as the equation of motion:

\[ \frac{\delta S}{\delta h_{\mu\nu}} = 0 \implies \Box h_{\mu\nu} = 0 \]  

(159)

This result will be our starting point, however consistently with our notation, we will express \( h_{\mu\nu} \) using the geometric algebra notation:

\[ \frac{1}{2} \Box(h_\mu h_\nu + h_\nu h_\mu) \frac{\delta h_{\mu\nu}}{\delta h_\mu} = 0 \]  

(160)

where the additional variation \( \frac{\delta h_{\mu\nu}}{\delta h_\mu} \) results from the chain rule: \( \frac{\delta S}{\delta h_\mu} = \frac{\delta S}{\delta h_{\mu\nu}} \frac{\delta h_{\mu\nu}}{\delta h_\mu} \).

The expression for \( \frac{\delta h_{\mu\nu}}{\delta h_\mu} \) is:

\[ \frac{\delta h_{\mu\nu}}{\delta h_\mu} = \frac{1}{\delta h_\mu} \frac{1}{2}(\delta(h_\mu h_\nu) + \delta(h_\nu h_\mu)) \]  

(161)

\[ = \frac{1}{\delta h_\mu} \frac{1}{2}(\delta(h_\mu) h_\nu + h_\mu \delta(h_\nu) + \delta(h_\nu) h_\mu + h_\nu \delta(h_\mu)) \]  

(162)

\[ = h_\nu \]  

(163)
Consequently, the equation of motion of $h_\mu$ and $h_\nu$ are, respectively:

\[
\frac{1}{2}\Box(h_\mu h_\nu + h_\nu h_\mu) = 0 \tag{164} \]

\[
\frac{1}{2}\Box(h_\mu h_\nu + h_\nu h_\mu) = 0 \tag{165} \]

We identify the solution by an ansatz:

\[
h_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left( \epsilon_\mu^{(\lambda)}(\vec{k})a_{\vec{k},\lambda} e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} + \epsilon_\mu^{(\lambda)*}(\vec{k})a_{\vec{k},\lambda}^* e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} \right) \tag{166} \]

\[
h_\nu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left( \epsilon_\nu^{(\lambda)}(\vec{k})a_{\vec{k},\lambda} e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} + \epsilon_\nu^{(\lambda)*}(\vec{k})a_{\vec{k},\lambda}^* e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} \right) \tag{167} \]

We promote $h_\mu(\vec{x},t)$ and $h_\nu(\vec{x},t)$ to operators:

\[
\hat{h}_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left( \epsilon_\mu^{(\lambda)}(\vec{k})\hat{a}_{\vec{k},\lambda} e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} + \epsilon_\mu^{(\lambda)*}(\vec{k})\hat{a}_{\vec{k},\lambda}^* e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} \right) \tag{168} \]

\[
\hat{h}_\nu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left( \epsilon_\nu^{(\lambda)}(\vec{k})\hat{a}_{\vec{k},\lambda} e^{i(\vec{k} \cdot \vec{x} - \omega_k t)} + \epsilon_\nu^{(\lambda)*}(\vec{k})\hat{a}_{\vec{k},\lambda}^* e^{-i(\vec{k} \cdot \vec{x} - \omega_k t)} \right) \tag{169} \]

### 2.4.4 Basis Vector x2 v. Metric Tensor

In the standard perturbative approach to gravity, the metric tensor $\hat{h}_{\mu\nu}$ is quantized, and the probabilities are calculated using the Born rule acting as follows: $\langle \psi | \hat{h}_{\mu\nu} | \psi \rangle$. In contrast, our approach quantizes the two basis vectors $\hat{h}_\mu$ and $\hat{h}_\nu$ separately and utilizes the double-copy product to obtain the probabilities as $\langle \psi | \hat{h}_\mu | \psi \rangle \langle \psi | \hat{h}_\nu | \psi \rangle$. In the latter case, the operators realize a double copy of the Dirac current, each requiring an individual application of the Born rule. The resulting probability calculation differs from the standard approach, as it involves the product of two separate Born rule applications. It further includes a SO(3,1)-valued transformation of the basis vectors embedded within the wavefunction and its evolution. It remains an open question whether this difference in probability calculations improves the divergence behavior of the resulting double-copy gravity theory compared to the standard perturbative approach with the Born rule.
2.5 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all dimensional configurations except those we have explored here (e.g. GA(0), GA(0, 1) and GA(3, 1)) are obstructed:

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Obstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>GA(0)</td>
<td>Unobstructed (\implies) statistical mechanics (170)</td>
</tr>
<tr>
<td>GA(0, 1)</td>
<td>Unobstructed (\implies) quantum mechanics (171)</td>
</tr>
<tr>
<td>GA(1, 0)</td>
<td>Negative probabilities in the RQM (172)</td>
</tr>
<tr>
<td>GA(2, 0)</td>
<td>No metric measurement (173)</td>
</tr>
<tr>
<td>GA(1, 1)</td>
<td>Negative probabilities in the RQM (174)</td>
</tr>
<tr>
<td>GA(0, 2)</td>
<td>Not isomorphic to a real matrix algebra (175)</td>
</tr>
<tr>
<td>GA(3, 0)</td>
<td>Not isomorphic to a real matrix algebra (176)</td>
</tr>
<tr>
<td>GA(2, 1)</td>
<td>Not isomorphic to a real matrix algebra (177)</td>
</tr>
<tr>
<td>GA(1, 2)</td>
<td>Not isomorphic to a real matrix algebra (178)</td>
</tr>
<tr>
<td>GA(0, 3)</td>
<td>Not isomorphic to a real matrix algebra (179)</td>
</tr>
<tr>
<td>GA(4, 0)</td>
<td>Not isomorphic to a real matrix algebra (180)</td>
</tr>
<tr>
<td>GA(3, 1)</td>
<td>Unobstructed (\implies) quantum gravity &amp; (SU(3) \times SU(2) \times U(1)) (181)</td>
</tr>
<tr>
<td>GA(2, 2)</td>
<td>Negative probabilities in the RQM (182)</td>
</tr>
<tr>
<td>GA(1, 3)</td>
<td>Not isomorphic to a real matrix algebra (183)</td>
</tr>
<tr>
<td>GA(0, 4)</td>
<td>Not isomorphic to a real matrix algebra (184)</td>
</tr>
<tr>
<td>GA(5, 0)</td>
<td>Not isomorphic to a real matrix algebra (185)</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>GA(6, 0)</td>
<td>No probability measure as a self-product (186)</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(\infty)</td>
<td>(\infty) (187)</td>
</tr>
</tbody>
</table>

Let us now demonstrate the obstructions mentioned above.

**Theorem 17** (Not isomorphic to a real matrix algebra). The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.
Proof. These geometric algebras are classified as follows:

\[
\begin{align*}
\text{GA}(0, 2) & \cong \mathbb{H} & (188) \\
\text{GA}(3, 0) & \cong \mathbb{M}_2(\mathbb{C}) & (189) \\
\text{GA}(2, 1) & \cong \mathbb{M}_2^2(\mathbb{R}) & (190) \\
\text{GA}(1, 2) & \cong \mathbb{M}_2(\mathbb{C}) & (191) \\
\text{GA}(0, 3) & \cong \mathbb{H} \oplus \mathbb{R} & (192) \\
\text{GA}(4, 0) & \cong \mathbb{M}_2(\mathbb{H}) & (193) \\
\text{GA}(1, 3) & \cong \mathbb{M}_2(\mathbb{H}) & (194) \\
\text{GA}(0, 4) & \cong \mathbb{M}_2(\mathbb{H}) & (195) \\
\text{GA}(5, 0) & \cong \mathbb{M}_2^2(\mathbb{H}) & (196)
\end{align*}
\]

The determinant of these objects is valued in \( \mathbb{C} \) or in \( \mathbb{H} \), where \( \mathbb{C} \) are the complex numbers, and where \( \mathbb{H} \) are the quaternions. \( \square \)

**Theorem 18** (Negative Probabilities in the RQM). *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. This category contains three dimensional configurations:

**GA(1, 0):** Let \( \psi(q) = a + be_1 \), then:

\[
(a + be_1)^3(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2 e_1 e_1 = a^2 - b^2
\]

which is valued in \( \mathbb{R} \).

**GA(1, 1):** Let \( \psi(q) = a + be_0 e_1 \), then:

\[
(a + be_0 e_1)^3(a + be_0 e_1) = (a - be_0 e_1)(a + be_0 e_1) = a^2 - b^2 e_0 e_1 e_0 e_1 = a^2 - b^2
\]

which is valued in \( \mathbb{R} \).

**GA(2, 2):** Let \( \psi(q) = a + be_0 e_0 e_1 e_2 \), where \( e_0^2 = -1, e_0^2 = -1, e_1^2 = 1, e_2^2 = 1 \), then:

\[
|\langle a + b \rangle^3(a + b)|_{3, 4}(a + b)_{3, 4}(a + b) = a^2 + 2ab + b^2
\]

(199)

(200)

We note that \( b^2 = b^2 e_0 e_0 e_1 e_2 e_0 e_0 e_1 e_2 = b^2 \), therefore:

\[
= (a^2 + b^2 - 2ab)(a^2 + b^2 + 2ab)
\]

(201)

\[
= (a^2 + b^2)^2 - 4a^2 b^2
\]

(202)

\[
= (a^2 + b^2)^2 - 4a^2 b^2
\]

(203)

which is valued in \( \mathbb{R} \).
In all of these cases the RQM probability can be negative.

We repeat the following self-products\cite{8} (Definition 17), which will help us demonstrate the next theorem:

$$GA(0,1) : \varphi \dagger \varphi$$ (204)
$$GA(2,0) : \varphi \dagger \varphi$$ (205)
$$GA(3,0) : [\varphi \dagger \varphi]_3 \varphi \dagger \varphi$$ (206)
$$GA(3,1) : [\varphi \dagger \varphi]_{3,4} \varphi \dagger \varphi$$ (207)
$$GA(4,1) : ([\varphi \dagger \varphi]_{3,4} \varphi \dagger \varphi)^\dagger ([\varphi \dagger \varphi]_{3,4} \varphi \dagger \varphi)$$ (208)

**Theorem 19** (No Metric Measurements). *This obstruction applies to GA(2,0). The probability measure of at least four self-products are required for the theory to be observationally complete with respect to the geometry.*

**Proof.** A metric measurement requires a probability measure of 4 self products because the metric tensor is defined using 2 self-products of the gamma matrices:

$$g_{\mu\nu} = \frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu)$$ (209)

Each pair of wavefunction products fixes one basis elements. Thus, two pairs of wavefunction products are required to fix the geometry from the wavefunction. As probability measures of four self-products begin to appear in 3D, then the GA(2,0) cannot produce a metric measurement as a quantum observable, thus its geometry is not observationally complete.

**Conjecture 1** (No probability measures as a self-product (in 6D)). *The multivector representation of the norm in 6D cannot satisfy any observables.*

**Argument.** In six dimensions and above, the self-product patterns found in Definition 17 collapse. The research by Acus et al.\cite{12} in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5 (f_4 (B) f_3 (f_2 (B) f_1 (B))) + b_2 B g_5 (g_4 (B) g_3 (g_2 (B) g_1 (B)))$$ (210)

This equation is not a multivector self-product but a linear sum of two multivector self-products\cite{12}. The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^3 - 2a_0^2 a_{17} + b_2 a_5^2 a_{47} p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0$$ (211)
$$b_1 a_0^3 a_{52} + 2b_2 a_5 a_{27} a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0$$ (212)
$$\langle 74 \text{ monomials} \rangle = 0$$ (213)
$$\langle 74 \text{ monomials} \rangle = 0$$ (214)
From Equation 210, it is possible to see that no observable $O$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

\[ b_1 O f_5(f_4(B)f_3(B)f_1(B)) + b_2 B g_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1 B f_5(f_4(B)f_3(B)f_1(B)) + b_2 O B g_5(g_4(B)g_3(g_2(B)g_1(B))) \]  

Any equality of the above type between $b_1 O$ and $b_2 O$ is frustrated by the factors $b_1$ and $b_2$, forcing $O = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory quantum theory (with observables) be constructible in 6D.

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

**Conjecture 2 (No probability measures as a self-product (above 6D)).** The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out GA(1, 3) even if in relativity, the signature of the metric $(+, -, -, -)$ versus $(-, -, -, +)$ does not influence the physics. However, in geometric algebra, GA(1, 3) represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued “probabilities” (i.e. $GA(1, 3) \cong M_2(\mathbb{H})$ and $\det M_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, 3+1D is the only dimensional configuration (other than the “non-geometric” configurations of $GA(0) \cong \mathbb{R}$ and $GA(0, 1) \cong \mathbb{C}$) in which a ‘least biased’ solution to the problem of maximizing the Shannon entropy of quantum measurements relative to an initial preparation, exists. This is an extremely constraining result regarding the possible spacetime configurations of the universe, and our ability (or inability) to construct a least biased theory to explain it.

### 3 Discussion

The principle of maximum entropy[3] states that the probability measure that best represents the current state of knowledge about a system is the one with the largest entropy, constrained by prior data.

In QM, an experiment begins with an initial preparation, followed by some transformations, and concludes with a final measurement of the system, yielding
the result of the experiment. Consistent with the maximum entropy principle, our aim is to derive the 'least biased' theory that connects the initial preparation \( \rho(q) \) to its final measurement \( \rho(q) \), thereby formulating the theory as a solution to a maximization problem, rather than merely by axiomatic stipulation.

Using this methodology, fundamental physics can be formulated as the general solution to a maximization problem involving the Shannon entropy of all possible measurements of an arbitrary system relative to its initial preparation, under the constraint of a vanishing phase. As such, the structure of the inferred theory is determined by the nature and generality of the employed constraint. In this paper, we have investigated these four entropy maximization problems:

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Vanishing Phase</th>
<th>Inferred Theory</th>
<th>Wavefunction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{E} = \sum_{q \in \mathbb{Q}} \rho(q)E(q) )</td>
<td>none</td>
<td>SM</td>
<td>N.A.</td>
</tr>
<tr>
<td>0 = \text{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 &amp; -E(q) \ E(q) &amp; 0 \end{bmatrix}</td>
<td>U(1)</td>
<td>QM</td>
<td>( \mathbb{C}^n )</td>
</tr>
<tr>
<td>0 = \frac{1}{2} \text{tr} \sum_{q \in \mathbb{Q}} \rho(q)M(q)</td>
<td>\text{Spin}^c(3, 1)</td>
<td>RQM</td>
<td>( \mathbb{R} \times \text{Spin}^c(3, 1) )^n</td>
</tr>
<tr>
<td>0 = \frac{1}{2} \text{tr} \int_{\mathcal{M}} \rho(x) \frac{M(x)}{n(x)} \sqrt{-</td>
<td>g</td>
<td>} \text{d}^4x \quad \text{Spin}^c(3, 1)</td>
<td>QG</td>
</tr>
</tbody>
</table>

where \( n = |\mathbb{Q}| \), denoting the length of the set \( \mathbb{Q} \).

Despite the differences in constraints, the four theories here-so formulated share a common logical genesis, adhere to the same principle of maximum entropy, and qualify as the least biased theory for their given constraint.

### 3.1 Interpretation

The Born rule is the least biased probability measure operating on a complex Hilbert space (Theorem 2). However, when extended to 3+1D, this is no longer the case. The least biased probability measure becomes the double-copy product (Theorem 3).

### 4 Conclusion

In conclusion, this paper presents a novel approach to quantum theory construction by solving a maximization problem on the Shannon entropy of all possible measurements of a system relative to its initial preparation, and under the constraint of a vanishing phase. By appropriately selecting the group of the vanishing phase, the solution resolves to quantum mechanics, relativistic quantum mechanics, or a candidate for a theory of quantum gravity. Our findings reveal the exceptional ability of this approach to generate theories that generalizes quantum probabilities through the introduction of vanishing phases. The
resulting measure is invariant under a wide range of geometric transformations, including those generated by the gauge groups of the Standard Model, and leads to the metric tensor as an operator involving a double copy of Dirac currents, without the need for additional assumptions beyond the vanishing phase. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for its specificity. By reducing fundamental physics to the optimal solution to an entropy maximization problem, the framework integrates statistical mechanics, quantum mechanics, relativistic quantum mechanics, and the metric operator, while also accounting for the dimensionality of spacetime and the gauge symmetries of particle physics, under a singular entropy maximization problem.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

- Data Availability Statement: No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

A SM

Here, we solve the Lagrange multiplier equation of SM.

\[
\mathcal{L}(\rho, \lambda, \beta) = -k_B \sum_{q \in Q} \rho(q) \ln \rho(q) + \lambda \left( 1 - \sum_{q \in Q} \rho(q) \right) + \beta \left( E - \sum_{q \in Q} \rho(q) E(q) \right)
\]

(216)
We solve the maximization problem as follows:

\[
\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho(q)} = 0 = -\ln \rho(q) - 1 - \lambda - \beta E(q) \quad (217)
\]

\[
0 = \ln \rho(q) + 1 + \lambda + \beta E(q) \quad (218)
\]

\[
\Rightarrow \ln \rho(q) = -1 - \lambda - \beta E(q) \quad (219)
\]

\[
\Rightarrow \rho(q) = \exp(-1 - \lambda) \exp(-\beta E(q)) \quad (220)
\]

\[
= \frac{1}{Z(\tau)} \exp(-\beta E(q)) \quad (221)
\]

The partition function, is obtained as follows:

\[
1 = \sum_{r \in \mathcal{Q}} \exp(-1 - \lambda) \exp(-\beta E(q)) \quad (222)
\]

\[
\Rightarrow (\exp(-1 - \lambda))^{-1} = \sum_{r \in \mathcal{Q}} \exp(-\beta E(q)) \quad (223)
\]

\[
Z(\tau) := \sum_{r \in \mathcal{Q}} \exp(-\beta E(q)) \quad (224)
\]

Finally, the probability measure is:

\[
\rho(q) = \frac{1}{\sum_{r \in \mathcal{Q}} \exp(-\beta E(q))} \exp(-\beta E(q)) \quad (225)
\]

**B  RQM in 3+1D**

\[
\mathcal{L}(\rho, \lambda, \tau) = -\sum_{q \in \mathcal{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q \in \mathcal{Q}} \rho(q)\right) + \zeta \left(-\frac{1}{2} \sum_{q \in \mathcal{Q}} \rho(q) \mathbf{M}_u(q)_{a \rightarrow 0, \mathbf{x} \rightarrow 0, \mathbf{b} \rightarrow 0}\right)
\]

Relative Shannon Entropy \hspace{1cm} Normalization Constraint \hspace{1cm} Vanishing Relativistic-Phase Anti-Constraint

(226)

The solution is obtained using the same step-by-step process as the 2D case, and yields:

\[
\rho(q) = \frac{1}{\sum_{r \in \mathcal{Q}} p(r) \det \exp(-\zeta \frac{1}{2} \mathbf{M}_u(r)_{a \rightarrow 0, \mathbf{x} \rightarrow 0, \mathbf{b} \rightarrow 0}) \det \exp(-\zeta \frac{1}{2} \mathbf{M}_u(q)_{a \rightarrow 0, \mathbf{x} \rightarrow 0, \mathbf{b} \rightarrow 0})} p(q)
\]

Spin\(^+\)(3,1) Invariant Ensemble \hspace{1cm} Spin\(^+\)(3,1) Born Rule

Initial Preparation

(227)
Proof. The Lagrange multiplier equation can be solved as follows:

\[
\frac{\partial \mathcal{L}(\rho, \lambda, \zeta)}{\partial \rho(q)} = 0 = -\ln \frac{\rho(q)}{p(q)} - p(q) - \lambda - \zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \tag{228}
\]

\[
0 = \ln \frac{\rho(q)}{p(q)} + p(q) + \lambda + \zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \tag{229}
\]

\[
\Rightarrow \ln \frac{\rho(q)}{p(q)} = -p(q) - \lambda - \zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \tag{230}
\]

\[
\Rightarrow \rho(q) = p(q) \exp(-p(q) - \lambda) \exp \left( -\zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \right) \tag{231}
\]

\[
= \frac{1}{Z(\zeta)} p(q) \exp \left( -\zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \right) \tag{232}
\]

The partition function \( Z(\zeta) \), serving as a normalization constant, is determined as follows:

\[
1 = \sum_{r \in Q} p(r) \exp(-p(q) - \lambda) \exp \left( -\zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \right) \tag{233}
\]

\[
\Rightarrow (\exp(-p(q) - \lambda))^{-1} = \sum_{r \in Q} p(r) \exp \left( -\zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \right) \tag{234}
\]

\[
Z(\zeta) := \sum_{r \in Q} p(r) \exp \left( -\zeta \text{tr} \frac{1}{2} M_u(q)|_{a \to 0, x \to 0, b \to 0} \right) \tag{235}
\]

\[
\square
\]

C SageMath program showing \([u^\dagger u]_{3,4}^\dagger u = \det M_u\)

```python
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()
```

35
Define the scalar variables for each basis element

```
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')
```

Create a general multivector

```
udegree0 = a
udegree1 = t*e0 + x*e1 + y*e2 + z*e3
udegree2 = f01*e0*e1 + f02*e0*e2 + f03*e0*e3 + f12*e1*e2 + f13*e1*e3 + f23*e2*e3
udegree3 = v*e0*e1*e2 + w*e0*e1*e3 + q*e0*e2*e3 + p*e1*e2*e3
udegree4 = b*e0*e1*e2*e3
u = udegree0 + udegree1 + udegree2 + udegree3 + udegree4
```

```
u2 = u.clifford_conjugate() * u
```

```
u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0 + u2degree1 + u2degree2 - u2degree3 - u2degree4
```

I = Matrix(SR, [[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]])

# MAJORANA MATRICES

```
y0 = Matrix(SR, [[0, 0, 0, 1], [0, 0, -1, 0], [0, 1, 0, 0], [-1, 0, 0, 0]])
y1 = Matrix(SR, [[0, -1, 0, 0], [-1, 0, 0, 0], [0, 0, 0, -1], [0, 0, -1, 0]])
y2 = Matrix(SR, [[0, 0, 0, 1], [0, 0, -1, 0], [0, -1, 0, 0], [1, 0, 0, 0]])
```
\[ y3 = \text{Matrix}(SR, \begin{bmatrix} -1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, -1, 0 \\ 0, 0, 0, 1 \end{bmatrix}) \]

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())

The program outputs

True

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector \( u \) is equal to the double-copy form: \( \det M_u = [u^\dagger u]_{3,4} u^\dagger u. \)

References


