Abstract

We show how to use rotations of vectors in GA to solve the following problem: “The following are known about a triangle: The ratio of the lengths of two sides; the angle formed by those sides; and the length of that angle’s bisector. Find the length of the side opposite that angle.”

Given \(\alpha, d,\) and \(k\) (but not \(l\)), find the length \(BC\).
1 Statement of the Problem

The problem that we solve here is a generalization of a contest problem that the YouTube channel “Mind Your Decisions” solved by conventional means at https://www.youtube.com/watch?v=BeuLmUjFFsk. We state that generalization as: Given $\alpha$, $d$, and $k$ (but not $l$), find the length $BC$ (Fig. 1).

2 Ideas that We Will Use

1. For any two parallel vectors $u$ and $v$, $u \wedge v = 0$.

2. For any two vectors $u$ and $v$, $u \wedge v = \langle uv \rangle_2$.

3. Ideas concern the rotation of a vector $v$ that is parallel to a given bivector $i$:

   (a) The vectors $ve^{i\theta}$ and $ve^{-i\theta}$ are rotations of $v$ by the same angle $\theta$, but in opposite directions.

   (b) For any angle $\theta$, and any vector $v$ that is parallel to bivector $i$, $ve^{i\theta} = e^{-i\theta}v$. Here is a proof:
Figure 2: Formulation of the problem in terms of vectors. Note that $b = kl\hat{e}^{-i\alpha}$; $d = \hat{d}\hat{d}$; and $c = ld\hat{e}^{i\alpha}$.

\begin{align*}
ve^{i\theta} &= v[\cos \theta + i \sin \theta] \\
&= v \cos \theta + vi \sin \theta \\
&= v \cos \theta - iv \sin \theta \\
&= [\cos \theta - i \sin \theta]v \\
&= e^{-i\theta}v.
\end{align*}

(c) From $ve^{i\theta} = e^{-i\theta}v$, we can see that $ve^{i\theta}v = v^2e^{-i\theta}$, and that $\hat{v}e^{i\theta}\hat{v} = e^{-i\theta}$.

3 Solution Strategy

We will use rotation of vectors to determine $l$, then (knowing $l$) we will use the Law of Cosines to calculate $BC$.

4 Formulation in Terms of Vectors

We formulate the problem as shown in Fig. 2.
5 Solution

We begin by recognizing that \([c - d] \parallel [d - b]\). Therefore, \([c - d] \wedge [d - b] = 0\), and

\[\langle [c - d] [d - b] \rangle_2 = 0\]

\[\langle [\hat{d} e^{-i\alpha} - \hat{d}] [d - k\hat{d} e^{-i\alpha}] \rangle_2 = 0\]

\[\langle [l\hat{d} e^{-i\alpha} \hat{d} - kl\hat{d} e^{-i\alpha} - d^2 + kd e^{-i\alpha}] \rangle_2 = 0\]

\[\langle [l\hat{d} e^{-i\alpha} - kl^2 e^{-i\alpha} - d^2 + kd e^{-i\alpha}] \rangle_2 = 0\]

\[\langle [d e^{-i\alpha} - k^2 e^{-i\alpha} - d^2 + kd e^{-i\alpha}] \rangle_2 = 0\]

\[-d \sin \alpha + kl \sin 2\alpha - kd \sin \alpha = 0\]

\[\therefore l = \frac{(k + 1) d \sin \alpha}{k \sin 2\alpha}\]

Because \(\sin 2\alpha = 2 \sin \alpha \cos \alpha\),

\[l = \frac{(k + 1) d \sin \alpha}{2k \cos \alpha}\]

Finally, from the Law of Cosines,

\[BC = \sqrt{AB^2 + AC^2 - 2 (AB) (AC) \cos 2\alpha}\]

\[= \sqrt{(kl)^2 + l^2 - 2 kl^2 \cos 2\alpha}\]

\[= l\sqrt{k^2 + 1 - 2k \cos 2\alpha}\]

\[= \left[\frac{(k + 1) d \sin \alpha}{2k \cos \alpha}\right] \sqrt{(k + 1)^2 - 2k (1 + \cos 2\alpha)}\]

\[= \left[\frac{(k + 1) d \sin \alpha}{2k \cos \alpha}\right] \sqrt{(k + 1)^2 - 4k \cos^2 \alpha}\]

References