The problem of the «negative frequencies» of the solutions of the D’Alembert equation
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Abstract

The appearance of solutions with negative frequency in the D’Alembert wave equation can be removed with a change of variable. The corresponding positive frequencies describe waves propagating from the "future" towards the "past". This argument was developed in the 1940s by the Italian mathematician Luigi Fantappiè [1] in the analysis of the solutions of the D’Alembert equation, but also of the Klein-Gordon equation (quantum particles of spin 0) and the Dirac equation (spin 1/2 particles).

1 The D’Alembert equation

As is known, the D’Alembert wave equation

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \quad (1)$$

is a linear, second-order partial differential equation (PDE) in $$\psi(x,y,z,t)$$. It is often written as:

$$\Box^2 \psi = 0,$$

where

$$\Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2},$$

is the Delambertian. The solutions of (1) are classified into: A) plane waves; B) spherical waves; C) standing waves. We are interested in case A. For the remaining cases, please refer to [2].

Rammentiamo che a differenza delle equazioni differenziali ordinarie (ODE), nelle PDE non interessa l’integrale generale, ma soluzioni soddisfacenti particolari condizioni al contorno o iniziali.

Given this, plane waves (described by a wave function $$\psi(x,y,z,t)$$) are characterized by a constant propagation direction verifying the following property: on every plane normal to this direction, the wave function $$\psi$$ depends only on the variable $$t$$. It follows that by orienting the x axis in the direction of propagation, the (1) is rewritten:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \quad (2)$$

**Definition 1** We say solution of the (2) any $$\psi \in C^2(\mathbb{R}^2)$$ which verifies (2).

**Notation 2** The definition (1) can be weakened by incorporating any finite discontinuities of the derivatives of $$\psi$$ and of $$\psi$$ itself.

**Theorem 3** A necessary and sufficient condition for $$\psi \in C^2(\mathbb{R}^2)$$ to be a solution of (2), is that it admits a decomposition of the type:

$$\psi(x,t) = f(x - ct) + g(x + ct), \quad f, g \in C^2(\mathbb{R}) \quad (3)$$
Proof. The sufficiency of the condition is immediate, since \( f(x - ct) \) and \( g(x - ct) \) are manifestly solutions of (2). To demonstrate the need, we perform the coordinate transformation in the \( xt \) plane:

\[
(x, t) \rightarrow (\xi, \eta),
\]

whose transformation equations are:

\[
\xi = x - ct, \quad \eta = x + ct,
\]

so that (4) is manifestly invertible:

\[
x = \frac{1}{2} (\xi + \eta), \quad t = \frac{1}{2c} (\eta - \xi)
\]

The (5) imply \( \psi(x, t) \equiv \psi[\xi(x, t), \eta(x, t)] \). Applying the derivation rule of composite functions:

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial x}
\]

From (5) \( \frac{\partial \xi}{\partial x} = 1, \frac{\partial \eta}{\partial x} = 1 \), so

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta},
\]

which can be rewritten as:

\[
\frac{\partial \psi}{\partial x} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \psi
\]

This relation is valid for every differentiable function \( \psi \). This circumstance suggests writing the formal expression

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},
\]

which links the partial differentiation operator with respect to \( x \), to the differentiation operators with respect to the variables \( \xi \) and \( \eta \). To determine the second partial derivative \( \frac{\partial^2 \psi}{\partial x^2} \), we can then write:

\[
\frac{\partial^2}{\partial x^2} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right)
\]

\[
= \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}
\]

It is clear that we can write

\[
\frac{\partial^2}{\partial \xi \partial \eta} = \frac{\partial^2}{\partial \eta \partial \xi}
\]

if and only if this operator acts on a function that verifies the hypotheses of Schwarz’s theorem on the invertibility of partial differentiation, i.e. of class \( C^2 \) on an assigned field \( A \) of \( \mathbb{R}^2 \). Since we are looking for solutions \( \psi \in C^2(\mathbb{R}^2) \), this condition is satisfied, so the (10) is written:

\[
\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}
\]

so

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2}
\]
Proceeding in the same way for the second derivative \( \frac{\partial^2 \psi}{\partial t^2} \)

\[
\frac{\partial^2 \psi}{\partial t^2} = c^2 \left( \frac{\partial^2 \psi}{\partial \eta^2} - 2 \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \xi^2} \right)
\] (13)

At this point we impose that \( \psi \) is a solution of the D’Alembert equation:

\[
0 = \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial \eta^2} = 4 \frac{\partial^2 \psi}{\partial \xi \partial \eta}
\]
i.e.

\[
\frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0,
\] (14)

which is the D’Alembert equation written in coordinates \((\xi, \eta)\), and integrates immediately. Indeed:

\[
\frac{\partial}{\partial \eta} \left( \frac{\partial \psi}{\partial \xi} \right) = 0 \implies \frac{\partial \psi}{\partial \xi} = \theta (\xi),
\]

being \( \theta (\xi) \in C^2 (\mathbb{R}) \) an arbitrary function. Integrating again:

\[
\psi (\xi, \eta) = \int \theta (\xi) \, d\xi + g (\eta),
\] (15)

where the arbitrary function \( g (\eta) \in C^2 (\mathbb{R}) \) plays the role of ”constant” of integration (with respect to the variable \( \xi \)). We therefore set:

\[
f (\xi) \overset{\text{def}}{=} \int \theta (\xi) \, d\xi,
\]

so

\[
\psi (\xi, \eta) = f (\xi) + g (\eta)
\]

By restoring the variables \((x, t)\) the statement follows. ■

**Definition 4** The solutions \( f(x - ct) \) and \( g(x + ct) \) are called **progressing wave** and **regressive wave**.

These names are suggested by the fact that taking time \( t \) as the real parameter, the graph of the function \( f(x - ct) \) [\( g(x + ct) \)] translates uniformly in the direction of the \( x \) axis and in the direction of the increasing [decreasing] abscissae. If \( t \) is the time, the translation occurs in both cases at speed \( c \), as shown in the Figures 1-2.

## 2 Fundamental solutions

**Fundamental solutions** are those for which \( \psi \) depends sinusoidally on \( x \pm ct \). They are called fundamental because from them we can reconstruct a more general solution by linear superposition (thanks to the linearity of (2)). For example:

\[
\psi (x, t) = A \cos \left[ \frac{2\pi}{\lambda} (x - ct) \right]
\] (16)
Figure 1: Progressive plane wave.

Figure 2: Regressive plane wave.
where \( A > 0 \) is the amplitude, while \( \lambda > 0 \) is the period of \( \psi \) with respect to \( x \) for a given instant. This quantity is called wavelength. Let’s define
\[
k \in \mathbb{R} \setminus \{0\} \mid |k| = \frac{2\pi}{\lambda}
\]
We call the positive real number \(|k|\) wavenumber. Continued
\[
\psi(x, t) = A \cos \left(\left|k\right| x - \omega t\right) \quad \text{(17)}
\]
having defined the angular frequency \( \omega = c |k| = \frac{2\pi}{T} \) where \( T \) is the period of the function \( \psi \) with respect to \( t \) (for an assigned \( x \)). If in \((17)\) we free ourselves from \(|k|\):
\[
\psi(x, t) = A \cos \left(k x - \omega t\right) \quad \text{(18)}
\]
which for \( k < 0 \) describes a regressive plane wave. Complex notation is preferable:
\[
\psi(x, t) = Ae^{i(kx - \omega t)} \quad \text{(19)}
\]

3 Solutions with negative frequency

The totality of \((19)\) does not exhaust the set of solutions of \((2)\) relative to the fundamental solutions. In fact, by imposing that \((19)\) is a solution, we have
\[
\omega^2 = c^2 k^2
\]
therefore negative frequencies are also allowed \( \omega = -ck < 0 \). In this case, the \((19)\) is rewritten
\[
\psi(x, t) = Ae^{i(kx + |\omega| t)} \quad \text{(20)}
\]
Performing the change of variable \( t' = -t \)
\[
\psi(x, t') = Ae^{i(kx - |\omega| t')} \quad \text{(21)}
\]
having
\[
-\infty < t = -t' < +\infty \implies +\infty > t' > -\infty \quad \text{(22)}
\]
It follows that while \( \psi(x, t) = Ae^{i(kx + |\omega| t)} \) describes the propagation of a plane wave with initial instant \( t_0 = -\infty \) («past») and with negative frequency, the function \( \psi(x, t') = Ae^{i(kx - |\omega| t')} \) describes the propagation of a plane wave with initial instant \( t'_0 = +\infty \) («future»), with positive frequency. This wave propagates backwards in time.
References
