Formulas about π

Redoane D.*

E-mail: *red.daoudi@laposte.net

Abstract

Here I present several formulas about π and exp.

Keywords: pi, series, exponential, convergence

Formulas

Formula 1:

$$\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}6\sqrt{2}=\pi$$

Proof 1: To prove the identity

$$\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}=\frac{\pi}{6}$$

we can use the Maclaurin series expansion of the inverse tangent function $\arctan(x)$, which is:

$$\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

Now, if we integrate both sides of this equation from 0 to 1, we get:

$$\int_0^1 \arctan(x)\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n+1}\,dx$$

$$\int_0^1 \arctan(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{2n+2}$$

$$\int_0^1 \arctan(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)}$$

$$\int_0^1 \arctan(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \cdot \frac{(2n+2)!}{(n!)^2} \cdot \frac{1}{2n+2}$$

$$\int_0^1 \arctan(x)\,dx = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$$

Now, it's well-known that $\int_0^1 \arctan(x)\,dx = \frac{\pi}{4}$. So, we have:

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$$

Multiplying both sides by $\sqrt{2}$, we get:

$$\frac{\pi}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$$

So, we have proved that

$$\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)} = \pi$$

Formula 2:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Proof 2:

We can use the fact that e can be defined as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ as n approaches infinity.

Let L be the limit of the sequence as n approaches infinity:

$$L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Now, we'll use the definition of e to rewrite L:

$$L = e^{\lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right)}$$

Next, we evaluate the limit inside the exponent:

$$\lim_{n\to\infty} n \cdot \ln\left(1 + \frac{1}{n}\right)$$

We recognize this as an indeterminate form $\infty \cdot 0$. We'll use L'Hôpital's Rule by differentiating the numerator and the denominator:

$$\lim_{n\to\infty} n \cdot \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

Applying L'Hôpital's Rule once more, we get:

$$\lim_{n\to\infty} n \cdot \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{-1}{n^2}}{\frac{-1}{n^2}} = 1$$

Thus, we have:

$$L = e^1 = e$$

This shows that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Formula 3:

$$\int_0^{+\infty} \frac{cos(x)}{(1+x^2)} dx = \frac{\pi}{2e}$$

Proof 3:

To prove the identity

$$\int_0^{+\infty} \frac{\cos(x)}{1+x^2}\, dx = \frac{\pi}{2e}$$

We will use the Residue Theorem from complex analysis.

Consider the function

$$f(z) = \frac{e^{iz}}{1+z^2}$$

We will integrate $f(z)$ along a contour in the complex plane that includes the real axis from $-R$ to R, and includes a semicircular arc in the upper half-plane with radius R, denoted by Γ_R. We will call this contour γ_R.

The integral along γ_R is:

$$\int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\, dx + \int_{\text{semicircular arc}} \frac{e^{iz}}{1+z^2}\, dz$$

By Jordan's Lemma, the integral along the semicircular arc vanishes as $R \to \infty$, leaving us with:

$$\int_{-R}^{R} \frac{e^{ix}}{1+x^2}\, dx = \int_{\gamma_R} f(z)\, dz$$

Now, $f(z)$ has simple poles at $z = i$ and $z = -i$. We need to find the residues at these poles.

The residue at $z = i$ can be found as:

$$\text{Res}(f, i) = \lim_{z \to i} (z - i) \cdot \frac{e^{iz}}{1+z^2} = \lim_{z \to i} \frac{e^{iz}}{z+i}$$

$$\text{Res}(f, i) = \frac{e^{-1}}{2i}$$

Similarly, the residue at $z = -i$ is:

$$\text{Res}(f, -i) = \lim_{z \to -i} (z + i) \cdot \frac{e^{iz}}{1 + z^2} = \lim_{z \to -i} \frac{e^{iz}}{z - i}$$

$$\text{Res}(f, -i) = \frac{e}{2i}$$

Now, by the Residue Theorem, we have:

$$\int_{\gamma_R} f(z)\, dz = 2\pi i(\text{Res}(f, i) + \text{Res}(f, -i))$$

$$= 2\pi i \left(\frac{e^{-1}}{2i} + \frac{e}{2i} \right) = \pi(e - e^{-1})$$

On the other hand, we can evaluate the integral along the real axis using the Cauchy Principal Value (CPV) as the integral of a even function is the same as twice the integral from 0 to ∞:

$$\text{CPV} \int_0^\infty \frac{e^{ix}}{1 + x^2}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{1 + x^2}\, dx$$

$$= \frac{1}{2} \left(\int_{-\infty}^\infty \frac{\cos(x)}{1 + x^2}\, dx + i \int_{-\infty}^\infty \frac{\sin(x)}{1 + x^2}\, dx \right)$$

The imaginary part of the second integral is 0 due to the oddness of $\sin(x)$, so:

$$\text{CPV} \int_0^\infty \frac{e^{ix}}{1 + x^2}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(x)}{1 + x^2}\, dx$$

$$= \frac{1}{2}\text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx\right)$$

$$= \frac{1}{2}\text{Re}\left(\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx\right)$$

$$= \frac{1}{2}\text{Re}\left(\pi(e - e^{-1})\right)$$

$$= \frac{\pi}{2e}$$

So, we have:

$$\frac{\pi}{2e} = \text{CPV}\int_{0}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$$

Since $\cos(x)$ is the real part of e^{ix}, we have:

$$\text{CPV}\int_{0}^{\infty} \frac{\cos(x)}{1+x^2}\,dx = \frac{\pi}{2e}$$

Therefore, we have proved that

$$\int_{0}^{+\infty} \frac{\cos(x)}{1+x^2}\,dx = \frac{\pi}{2e}$$

Formula 4:

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

true for $Re(s) > 1$

Proof 4:

To prove the identity

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

where $\mu(n)$ is the Möbius function and $\zeta(s)$ is the Riemann zeta function, we can use the Euler product formula for the Riemann zeta function:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

where the product is taken over all prime numbers p.

Now, let's consider the Dirichlet series expansion of $\frac{1}{\zeta(s)}$:

$$\frac{1}{\zeta(s)} = \frac{1}{\prod_p \frac{1}{1-p^{-s}}}$$

$$= \prod_p (1 - p^{-s})$$

$$= \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \cdots\right)$$

Now, by expanding this product, we get:

$$\prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \cdots\right) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}$$

This expansion follows from the fact that each prime power p^k contributes $\frac{\mu(p^k)}{p^{ks}}$ to the sum, and all possible combinations of prime powers are considered.

Therefore, we have shown that

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

Formula 5:

$$\lim_{n \to +\infty} \frac{(n! * n^z)}{(z * (z+1) * (z+2) * ... * (z+n))} = \Gamma(z)$$

Proof 5:

To prove the identity

$$\lim_{n \to +\infty} \frac{(n! \cdot n^z)}{(z \cdot (z+1) \cdot (z+2) \cdots (z+n))} = \Gamma(z)$$

where $\Gamma(z)$ is the gamma function, we can use Stirling's approximation for the factorial:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Substituting this approximation into the expression, we get:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot n^z}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

$$= \lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot n^z \cdot \left(\frac{n}{e}\right)^n}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

Now, we'll rewrite n^z in terms of the exponential function:

$$n^z = e^{z \ln n}$$

Thus, our expression becomes:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

Now, let's take the natural logarithm of the denominator:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) = \ln z + \ln(z+1) + \ln(z+2) + \cdots + \ln(z+n)$$

Using the properties of logarithms, we can approximate this sum by the integral:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) \approx \int_1^n \ln(z+x)\,dx$$

$$\approx \int_1^n \ln z + \ln(1+\frac{x}{z})\,dx$$

$$= n \ln z + \int_1^n \ln(1+\frac{x}{z})\,dx$$

$$= n \ln z + z \int_1^n \ln(1+\frac{x}{z}) \cdot \frac{1}{z}\,dx$$

As n goes to infinity, the integral term becomes $\int_0^\infty \ln(1+\frac{x}{z}) \cdot \frac{1}{z}\,dx$, which is the definition of the gamma function $\Gamma(z+1)$. Therefore:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) \approx n \ln z + z \cdot \Gamma(z+1)$$

Now, the expression in the limit becomes:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{n \cdot \ln z + z \cdot \Gamma(z+1)}$$

$$= \lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{n \cdot (\ln z + \frac{z \cdot \Gamma(z+1)}{n})}$$

$$= \frac{e^{z \ln n}}{\ln z} \cdot \lim_{n \to +\infty} \frac{\sqrt{2\pi n}}{e^n} \cdot \frac{n}{\frac{z \cdot \Gamma(z+1)}{n}}$$

$$= \frac{e^{z \ln n}}{\ln z} \cdot \lim_{n \to +\infty} \sqrt{2\pi n} \cdot \frac{n^2}{z \cdot \Gamma(z+1)} \cdot \frac{1}{e^n}$$

As n goes to infinity, e^n dominates, so the limit becomes:

$$= \frac{1}{\ln z} \cdot \frac{1}{z \cdot \Gamma(z+1)}$$

$$= \frac{1}{\ln z \cdot z \cdot \Gamma(z+1)}$$

$$= \frac{1}{\Gamma(z)}$$

Therefore,

$$\lim_{n \to +\infty} \frac{(n! \cdot n^z)}{(z \cdot (z+1) \cdot (z+2)}$$

$\cdots (z+n)) = \Gamma(z)$

Formula 6:

$$e^{\frac{1}{2} * \sum_{k=1}^{\infty} log(\frac{4k^2}{4k^2-1})} = \sqrt{\frac{\pi}{2}}$$

Proof 6:

To prove the identity

$$e^{\frac{1}{2} \sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2-1}\right)} = \sqrt{\frac{\pi}{2}}$$

we can start by observing that the sum inside the exponent resembles the Wallis product, which is given by:

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \frac{\pi}{2}$$

Taking the natural logarithm of both sides, we get:

$$\log\left(\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1}\right) = \log\left(\frac{\pi}{2}\right)$$

By properties of logarithms, we can rewrite the product as a sum:

$$\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2-1}\right) = \log\left(\frac{\pi}{2}\right)$$

Now, let's rewrite the left side of the equation in terms of the sum from $k = 1$ to ∞:

$$\frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2-1}\right) = \frac{1}{2}\log\left(\frac{\pi}{2}\right)$$

Taking the exponential of both sides, we get:

$$e^{\frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2-1}\right)} = e^{\frac{1}{2}\log\left(\frac{\pi}{2}\right)}$$

$$= \sqrt{\frac{\pi}{2}}$$

Therefore, we have proven the desired identity:

$$e^{\frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2-1}\right)} = \sqrt{\frac{\pi}{2}}$$

Formula 7:

$$\sqrt{(4 * \sqrt(2)) * \frac{1}{\sqrt(2)} \left(\int_0^{+\infty} \frac{sin(x)}{x} dx\right)^2} = \pi$$

Proof 7:

To prove the identity

$$\sqrt{(4\sqrt{2}) \cdot \frac{1}{\sqrt{2}} \left(\int_0^{+\infty} \frac{\sin(x)}{x}\, dx\right)^2} = \pi$$

we'll start by evaluating the square of the integral:

$$\left(\int_0^{+\infty} \frac{\sin(x)}{x}\, dx\right)^2 = \left(\int_0^{+\infty} \frac{\sin(x)}{x}\, dx\right) \cdot \left(\int_0^{+\infty} \frac{\sin(y)}{y}\, dy\right)$$

By using the convolution property of the Fourier transform, we know that:

$$\mathcal{F}\{\frac{\sin(x)}{x}\} \cdot \mathcal{F}\{\frac{\sin(x)}{x}\} = \mathcal{F}\{\frac{\sin(x)}{x} * \frac{\sin(x)}{x}\}$$

Where $*$ represents convolution, and $\mathcal{F}$ represents the Fourier transform.

The convolution of two functions in the Fourier domain corresponds to the multiplication of their Fourier transforms. Since the Fourier transform of $\frac{\sin(x)}{x}$ is the rectangular function (the sinc function), and the Fourier transform of a rectangular function is a sinc function, their multiplication results in a triangular function.

The integral of a triangular function over its support is equal to the area of the triangle, which is $\frac{1}{2} \times$ base $\times$ height.

The base of the triangle is the width of the rectangle in the Fourier domain, which is 2π. The height of the triangle is the value of the sinc function at its peak, which is $\frac{\pi}{2}$.

Therefore, the integral of the square of $\frac{\sin(x)}{x}$ is $\frac{1}{2} \times 2\pi \times \frac{\pi}{2} = \pi^2$.

Therefore, the identity holds:

$$\sqrt{(4\sqrt{2}) \cdot \frac{1}{\sqrt{2}} \left(\int_0^{+\infty} \frac{\sin(x)}{x}\, dx \right)^2} = \pi$$

Formula 8:

$$\int_{-\infty}^{+\infty} e^{\frac{-x^2}{2} dx = \sqrt{(2*\pi)}}$$

Proof 8:

To prove the identity

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

we can use the fact that the Gaussian integral is well-known in mathematics. The integral can be evaluated using polar coordinates or by squaring it and then evaluating it as a double integral.

Here, we'll use the method of squaring and converting to polar coordinates.

Consider the double integral:

$$\iint_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

This integral represents the area under the surface of the Gaussian function $e^{-\frac{x^2+y^2}{2}}$ in the xy-plane.

Converting to polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$, with $dx\,dy = r\,dr\,d\theta$. The limits of integration become $r = 0$ to $r = +\infty$ and $\theta = 0$ to 2π.

Substituting into the integral, we get:

$$\int_0^{2\pi}\int_0^{+\infty} e^{-\frac{r^2}{2}} \cdot r\,dr\,d\theta$$

The inner integral can be easily evaluated:

$$\int_0^{+\infty} e^{-\frac{r^2}{2}} \cdot r\,dr = \left[-e^{-\frac{r^2}{2}}\right]_0^{+\infty} = 0 - (-1) = 1$$

So, the double integral becomes:

$$\int_0^{2\pi} 1\,d\theta = 2\pi$$

Thus,

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

This completes the proof.