CAN EINSTEIN TENSOR BE GENERALIZED?

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Abstract. In this short paper I will write a possible generalizations of Einstein tensor and energy momentum tensor that will lead to generalizations of Einstein field equations.
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1. Einstein tensor

Einstein tensor [1] is basis of General Relativity, it has vacuum solutions equal to:

\[ G^{\mu \nu} = 0 \]  

(1.1)

Another property is that it is symmetric and it’s covariant derivative is equal to zero from it follows that:

\[ \nabla_{\nu} G^{\mu \nu} = 0 \]  

(1.2)

\[ G^{\mu \nu} = G^{\nu \mu} \]  

(1.3)

It plays crucial role in Einstein field equations [2] as it is left side of field equation:

\[ G^{\mu \nu} = \kappa T^{\mu \nu} \]  

(1.4)

Where tensor itself is equal to:

\[ G^{\mu \nu} = R^{\mu \nu} - \frac{1}{2} R g^{\mu \nu} \]  

(1.5)

So field equations are equal to:

\[ R^{\mu \nu} - \frac{1}{2} R g^{\mu \nu} = \kappa T^{\mu \nu} \]  

(1.6)

But in whole paper I will be using not contravariant form but covariant form of this tensor so \( G_{\mu \nu} \). It will be same tensor but with covariant indexes, it will be equal to:

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \]  

(1.7)

So field equation is same but with covariant indexes:

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \kappa T_{\mu \nu} \]  

(1.8)

This is form of field equations I will use in whole paper.
2. Riemann tensor and generalized Einstein tensor

To build a generalized Einstein tensor I need to assume some kind of basis of deriving it. I will use Riemann tensor contractions are that basis, I want generalized tensor to have same contractions as Riemann tensor \[3\] \[4\]. It means that if I write Riemann tensor contractions they will be same as contractions of generalized Einstein tensor:

\[
\begin{align*}
  g^{\alpha\mu} R_{\alpha\mu\beta\nu} &= 0 \\
  g^{\alpha\beta} R_{\alpha\mu\beta\nu} &= R_{\mu\nu} \\
  g^{\alpha\nu} R_{\alpha\mu\beta\nu} &= -R_{\mu\beta} \\
  g^{\mu\beta} R_{\alpha\mu\beta\nu} &= -R_{\alpha\nu} \\
  g^{\mu\nu} R_{\alpha\mu\beta\nu} &= R_{\alpha\beta} \\
  g^{\beta\nu} R_{\alpha\mu\beta\nu} &= 0
\end{align*}
\]

(2.1) \hspace{2cm} (2.2) \hspace{2cm} (2.3) \hspace{2cm} (2.4) \hspace{2cm} (2.5) \hspace{2cm} (2.6)

So I can write down now same contractions but for generalized Einstein tensor \(G_{\alpha\mu\beta\nu}\):

\[
\begin{align*}
  g^{\alpha\mu} G_{\alpha\mu\beta\nu} &= 0 \\
  g^{\alpha\beta} G_{\alpha\mu\beta\nu} &= G_{\mu\nu} \\
  g^{\alpha\nu} G_{\alpha\mu\beta\nu} &= -G_{\mu\beta} \\
  g^{\mu\beta} G_{\alpha\mu\beta\nu} &= -G_{\alpha\nu} \\
  g^{\mu\nu} G_{\alpha\mu\beta\nu} &= G_{\alpha\beta} \\
  g^{\beta\nu} G_{\alpha\mu\beta\nu} &= 0
\end{align*}
\]

(2.7) \hspace{2cm} (2.8) \hspace{2cm} (2.9) \hspace{2cm} (2.10) \hspace{2cm} (2.11) \hspace{2cm} (2.12)

From it comes another part of equations that is generalized energy momentum tensor \[5\], that will have same contraction properties as Riemann tensor and generalized Einstein tensor to follow a field equation:

\[
\begin{align*}
  g^{\alpha\nu} T_{\alpha\mu\beta\nu} &= 0 \\
  g^{\alpha\beta} T_{\alpha\mu\beta\nu} &= T_{\mu\nu} \\
  g^{\alpha\nu} T_{\alpha\mu\beta\nu} &= -T_{\mu\beta} \\
  g^{\mu\beta} T_{\alpha\mu\beta\nu} &= -T_{\alpha\nu} \\
  g^{\mu\nu} T_{\alpha\mu\beta\nu} &= T_{\alpha\beta} \\
  g^{\beta\nu} T_{\alpha\mu\beta\nu} &= 0
\end{align*}
\]

(2.13) \hspace{2cm} (2.14) \hspace{2cm} (2.15) \hspace{2cm} (2.16) \hspace{2cm} (2.17) \hspace{2cm} (2.18)

So from it comes that generalized Einstein tensor reduces either to plus-minus Einstein tensor or zero and generalized energy momentum tensor have to obey same rule to make it consistent with field equations.
3. Generalized Einstein tensor

I will first write generalized Einstein tensor and generalized energy momentum tensor, then will show that they indeed follow contractions properties. So those tensors are equal to:

\[ G_{\alpha\mu\beta\nu} = 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \]  

(3.1)

\[ T_{\alpha\mu\beta\nu} = \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \]

(3.2)

\[ g^{\alpha\mu} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = 0 \]  

(3.3)

\[ g^{\alpha\beta} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = G_{\mu\nu} \]  

(3.4)

\[ g^{\alpha\nu} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = -G_{\mu\beta} \]  

(3.5)

\[ g^{\mu\beta} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = -G_{\alpha\nu} \]  

(3.6)

\[ g^{\alpha\nu} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = G_{\alpha\beta} \]  

(3.7)

\[ g^{\nu\beta} \left( 2R_{\alpha\mu\beta\nu} - \frac{1}{2} (R_{\alpha\beta g_{\mu\nu}} + R_{\mu\nu g_{\alpha\beta}}) + \frac{1}{2} (R_{\mu\beta g_{\alpha\nu}} + R_{\alpha\nu g_{\beta\mu}}) \right) = 0 \]  

(3.8)

\[ g^{\alpha\mu} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = 0 \]  

(3.9)

\[ g^{\alpha\beta} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = T_{\mu\nu} \]  

(3.10)

\[ g^{\alpha\nu} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = -T_{\mu\beta} \]  

(3.11)

\[ g^{\mu\beta} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = -T_{\alpha\nu} \]  

(3.12)

\[ g^{\alpha\nu} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = T_{\alpha\beta} \]  

(3.13)

\[ g^{\nu\beta} \left( \frac{1}{2} (T_{\alpha\beta g_{\mu\nu}} + T_{\mu\nu g_{\alpha\beta}}) - \frac{1}{2} (T_{\mu\beta g_{\alpha\nu}} + T_{\alpha\nu g_{\beta\mu}}) - \frac{1}{6} T (g_{\alpha\beta g_{\mu\nu}} - g_{\beta\mu g_{\alpha\nu}}) \right) = 0 \]  

(3.14)
In this short paper I showed possible generalization of Einstein tensor. This leads to generalized energy momentum tensor, that combined create a new field equation:

\[ G_{\alpha\beta\mu\nu} = \kappa T_{\alpha\mu\beta\nu} \] (4.1)

Contractions of this field equation lead to zero or plus-minus Einstein tensor. That gives new equation for space-time curvature and new vacuum equations that will be equal to:

\[ G_{\alpha\mu\beta\nu} = 0 \] (4.2)

Problem with this equation is that is really hard to solve, as its a four rank tensor. For example field equation will take form for simplest case of vacuum:

\[ 2R_{\alpha\beta\alpha\beta} - \frac{1}{2} (R_{\alpha\alpha}g_{\beta\beta} + R_{\beta\beta}g_{\alpha\alpha}) + \frac{1}{2} (R_{\alpha\beta}g_{\alpha\beta} + R_{\alpha\beta}g_{\beta\alpha}) = 0 \] (4.3)

From fact that independent components for Riemann tensor in case of spherical symmetric space-time are only six of them [6] and there are no cross terms for Ricci tensor I will get:

\[ 2R_{\alpha\beta\alpha\beta} - \frac{1}{2} (R_{\alpha\alpha}g_{\beta\beta} + R_{\beta\beta}g_{\alpha\alpha}) = 0 \] (4.4)

Where I can write independent components:

\[ 2R_{0101} - \frac{1}{2} (R_{00}g_{11} + R_{11}g_{00}) = 0 \] (4.5)
\[ 2R_{0202} - \frac{1}{2} (R_{00}g_{22} + R_{22}g_{00}) = 0 \] (4.6)
\[ 2R_{0303} - \frac{1}{2} (R_{00}g_{33} + R_{33}g_{00}) = 0 \] (4.7)
\[ 2R_{1212} - \frac{1}{2} (R_{11}g_{22} + R_{22}g_{11}) = 0 \] (4.8)
\[ 2R_{1313} - \frac{1}{2} (R_{11}g_{33} + R_{33}g_{11}) = 0 \] (4.9)
\[ 2R_{2323} - \frac{1}{2} (R_{22}g_{33} + R_{33}g_{22}) = 0 \] (4.10)

From it follows clearly that vacuum solutions have non-vanishing Ricci tensor, even in simplest case.
REFERENCES


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