PROOF OF COLLATZ CONJECTURE

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Abstract. Proof of this conjecture has been elusive for over 60 years. The key to a proof was to find the right combination of logic and equations to complete the proof. We section the Natural numbers into 3 mutually exclusive sets. We first assume that for the first set that there is a number in the set that does not obey Collatz and show this cannot be true since it leads to a contradiction. Using this result we show that the other 2 sets must also obey Collatz.

1. Introduction

1.1. **Collatz.** Long considered unsolvable the Collatz Conjecture is simple to understand but difficult to find an attack or method to prove, although believed it to be true. We show in this proof that it is indeed true. We first show a number of Lemmas that can be used to establish the proof.

2. Definitions

Definition 2.1. The Collatz Conjecture Function for $x \in \mathbb{N}$

$$f(x) = \begin{cases} \frac{x}{2} & x = even \\ \frac{3x+1}{2} & x = odd \end{cases}$$

when applied to a root, x, and subsequent results will always create a sequence that includes the number 1. We use $\frac{3x+1}{2}$ instead of $3x + 1$ since $3x + 1$ always results in an even number and we avoid one step of dividing by 2. We can think of the function as providing a sequence of elements of $\mathbb{N}$, as follows: f(x_0)=x_1, f(x_1)=x_2, f(x_2)=x_3, ..., x_n. Numbers that obey the Collatz Conjecture will have x_n=1 for some index value.

Definition 2.2. root This term is given to the initial value used to apply the Collatz function. The result is the first element of a sequence (index of 1 for the sequence. Applying the Collatz function to each prior result to form a sequence. If the number 1 is reached, the sequence will end since the sequence values would loop with 1 2 1 forever.

Definition 2.3. index n. This is the index for a sequence created by applying the Collatz function. It represents the position in the sequence with the root as position 0. There may be reasons to stop the sequence at some specified index value.

Theorem 2.4. *The Collatz Conjecture for $x \in \mathbb{N}$, the function*

$$f(x) = \begin{cases} \frac{x}{2} & x = even \\ \frac{3x+1}{2} & x = odd \end{cases}$$

when applied to a root $x \in \mathbb{N}$ and subsequent results will always create a sequence that includes the number 1. We use $\frac{3x+1}{2}$ instead of $3x + 1$ since $3x + 1$ always results in an even number and we avoid one step of dividing by 2.

Lemma 1. *If the Collatz function is applied to a root $x \in \mathbb{N}$ in order to create a sequence of numbers, x_1, x_2, x_3, . . ., x_n, that for some index, n, results in a sequence value of 1, then all of the numbers in the sequence are thereby proven to have a 1 in their sequence if any of these numbers were to be used as a starting root . This satisfies the requirements for both numbers to satisfy the requirements if the Collatz Conjecture. Thus all numbers in such a sequence would satisfy the conjecture if the root does.*

Proof. If we create, using the Collatz function, a sequence a starting with a root of x_0 that has for some index number n, a value of $x_n = 1$, and for an index m, between 0 and n, a value of x_m, then using x_m as root to create a sequence using the Collatz function, must produce the same sequence for index numbers 0 thru n-m as x_0 did for sequence numbers m thru n.This is because all of the same operations are performed from x_m to get a sequence element of 1 in both cases. □

Lemma 2. *Useful properties of odd number set A_5: For the odd number set $A_5 = \{x \mid x = 2^2 + 1 + 2^2 p\}$, and $p \in \mathbb{N}$, $\frac{3x+1}{2}$ is always an even number. Also, dividing that even number by 2 until an odd number results will produce an odd number smaller than the original number.*

Proof. Let $x_1 = 2^2 + 1 + 2^2 p$, $p \in \mathbb{N}$
$f(x_1) = \frac{3(2^2+1+2^2p)+1}{2}$
$= \frac{2^3+2^2+2^2+(2^3+2^2)p}{2}$
$= 2^2 + 2 + 2 + (2^2 + 2)p$
$= 2^3 + (2^2 + 2)p = x_2$ which can never be an odd number.
Dividing x_2 by 2 gives $x_3 = 2^2 + (2 + 1)p$
$\therefore x_3$ is less than x_2 is less than x_1 □

Lemma 3. *Useful properties of odd number set A_3: Define the odd number set $A_3 = \{x \mid x = 2^2 - 1 + 2^2 p\}$, $p \in \mathbb{N}$.*
The set A_3 can also be defined as $\{x \mid x = 2^i k_1 \; - 1\}$, k_1 is odd, $i \geq 2$. If $x \in A_3$, $\frac{3x+1}{2}$ never results in an even number.

Proof. Let $x = 2^2 - 1 + 2^2 p$, we calculate $\frac{3x+1}{2}$ as follows:
$\frac{2^3+2^2-3+1+(2^3+2^2)p}{2} = 2^2 + 2 + (2^2 + 2)p - 1$ which is always odd.
Also, $2^2 - 1 + 2^2 p = 2^2(1 + p) - 1$.
Let $1 + p = k_0$, then $2^2(1 + p) - 1 = 2^2 k_0 - 1$.
$\exists k_1$ and m, so that $k_0 = 2^m k_1$, and k_1 is odd,
$\therefore 2^2 k_0 = 2^i k_1$, where k_1 is odd and i=2+m □

Example 2.5. *(for example) Let p be any odd number, then $1+p$, is always an even number. If $p = 9$, then $2^2 - 1 + 2^2 p = 4 - 1 + 4(9) = 39$, $k_0 = (2)(5)$, so*
$k_1 = 5$, $m = 1$, $2^2 k_0 - 1 = 2^2(2)(5) - 1 = 39$, $i = 3$

Lemma 4. *For $x \in \mathbb{N}$, $3x + 1$, is always even.*

Proof. For $x, y \in \mathbb{N}$, let $x = 2y + 1$
$3x + 1 = 3(2y + 1) + 1 = 6y + 3 + 1 = 6y + 4$, which is always even. For this reason we can shorten the number of steps by using $\frac{3x+1}{2}$ instead of $3x + 1$ for odd numbers in defining the function of Collatz □

Lemma 5. *$\forall x \in A_3$ we may develop a formula for creating the values of any particular sequence number element for a sequence based on the initial x value to start and the sequence number. Such a formula can be used as long as subsequent elements are odd. The formula breaks after the first appearance of one odd number from A_5 and the next value which will be even. "Breaks" only means that the formula cannot be used for sequence number values beyond this point, but that does not matter since we have met a goal of finding a sequence value from A_5 The sequence number values would be seen as follows for the given sequence numbers. Where n is the sequence number $n \geq 3$*

$$f(x) = \frac{3^n(x) + 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 3(2^{n-2}) + 2^{n-1}}{2^n}$$

Proof. Choosing a starting x, $x \in A_3$ and applying Collatz to each result until an even number results will create a general formula for each each element of the Collatz sequence that starts with with this Integer.as shown below for the first 4 elements after thestating roor.

(1) $\frac{3x+1}{2}$
(2) $\frac{3^2(x)+3+2)}{2^2}$
(3) $\frac{3^3(x)+3^2+3(2)+2^2}{2^3}$

(4) $\frac{3^4(x)+3^3+3^2(2)+3(2^2)+2^3}{2^4}$

Continuing in this fashion results in the f(x) above for the n'th sequence value. This is assuming that there are no even numbers in the sequence until possibly the last one. If and when an even number is in the sequence, one can calculate no further with this equation since it is developed to only apply $\frac{3x+1}{2}$ to get sequence values. □

Lemma 6. *For $n \in \mathbb{N}$, 3^n can be expanded to a generalized form as follows*
$n \geq 3$
$3^n = 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 2^{n-1} + 2^n$

Proof.

$$\begin{aligned}3^n &= 3^{n-1} + 3^{n-1} + 3^{n-1} \\ &= 3^{n-1} + 3^{n-1}(2) \\ &= 3^{n-2}(6) + 3^{n-1} \\ &= 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(12) \\ &= 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + 3^{n-4}(24) \\ &= + \cdots + \\ &= 3^{n-1} + \cdots + 2^{n-1} + 2^{n-1} + 2^{n-1} \\ &= 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 2^{n-1} + 2^n\end{aligned}$$

Note that it is helpful to compare the last general formulas from Lemma 5 and this one visually.

$$3^n = 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 2^{n-1} + 2^n \tag{1}$$

$$f(x) = \frac{3^n(x) + 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 3(2^{n-2}) + 2^{n-1}}{2^n} \tag{2}$$

□

Lemma 7. *For $x = x_0 \in A_3$, as a root for a sequence using the Collatz function, $\frac{3x+1}{2}$, is in A_5, when p is even, then, $x_1 \in A_5$ and x_2 is an even number. And conversely, if an even number is at index 2 when starting with any odd root from A_3, then, $x_1 \in A_5$.*

Proof. Let $x = 2^2 - 1 + 2^2p$, let p=2t
$\frac{3x+1}{2} = \frac{2^2(3)-3+2^2(6t)+1}{2}$
$= \frac{2^2+2^2+2+2-2+2^2(6t)}{2}$
$= \frac{2^2+2^2+2+2^2(6t)}{2}$
$= \frac{2^3+2+2^2(6t)}{2} = 2^2 + 1 + 2^2(3t)$. Let 3t = c, then $3^2 + 1 + 2^2c$, which is one definition of A_5 By Lemma 3, above $\frac{3x+1}{2}$ never results in an even number when $x \in A_3$. □

Proof. We now use the above Lemmas to prove the Collatz Conjecture.

The first elements of $\mathbb{N}$ can be calculated and shown to obey the Conjecture. Assume $\exists x_0 \in A_5$, that does not satisfy Collatz. Then, since A_5 is an ordered set, there must be a smallest such number. But, by Lemma 2, if x_0 is the root f(x_0) results in a smaller odd number, x_s, as the next odd number in the sequence, which by Lemma 1 forces a contradiction. By assuming that x_s satisfies Collatz, we have assumed that if we use it as root to generate a sequence, the sequence will contain a 1. Since x_s is in the sequence for x_0, this forces a 1 into the sequence for x_0 as a root. Since the element of A_5 chosen for the root can be any number in A_5, all of A_5 satisfies Collatz by Lemma 1.

If there is to be an odd number that does not satisfy Collatz, then it must be found in A_3. For $x_0 \in A_3$ and $x = 2^2 - 1 + 2^2p$, consider two possibilities. For p as an even number, we saw from Lemma 7 that for the sequence at index position 1, $x_1 \in A_5$. From Lemma 1 and the fact that all numbers in A_5 obey Collatz, we conclude that these numbers from A_3 obey Collatz.

We now consider the remainder of the members of A_3, where p is odd. From Lemma 3, we know that members of A_3 never result in an even number when $\frac{3x+1}{2}$ is applied once. From Lemma 2 we see that applying $\frac{3x+1}{2}$ to a $x \in A_5$ results in an even number. Thus, if we apply the Collatz Function to start a sequence with $x_0 \in A_3$, and if we find at index, i, of the sequence, the first even number in the sequence, then the number at index i-1 is from A_5. Again, by Lemma 1, the chosen $x_0 \in A_3$ would be proven to obey Collatz. Our next goal then is to show that for any $x \in A_3$, we can find a first even number in the sequence. and we need to only look at index $i \geq 3$, when p is odd.

The equation 2, above, was developed with the assumption that applying Collatz starting with a root does not need to use any part of the Collatz function except $\frac{3x+1}{2}$. Since members of A_3 always produce an odd number for the sequence we can use this function to find an index where the first even number. If that index is i, then the sequence value at index i-1 must be a number from A_5 by Lemma 7.

If $x_0 = 2^2 - 1 + 2^2 p$ and p is an odd number, in order to find the first even number in a sequence with a root in A_3 we will let $x \in A_3$, and use the general function from equation 3, to show that every member of A_3 (that is defined with an odd p) can be shown to have an even number in the sequence in at least sequence index 3.

$$f(x) = \frac{3^n(x) + 3^{n-1} + 3^{n-2}(2) + 3^{n-3}(2^2) + \cdots + 3(2^{n-2}) + 2^{n-1}}{2^n} \quad (3)$$

Let $x = 2^i k_1 - 1$, as described in Lemma 3. Then to get the value at index n evaluate f(x). $f(x) = \frac{3^n(2^i k_1 - 1) + 3^{n-1}) + \cdots + 2^{n-1}}{2^n}$ Using equation 2 to eliminate terms in this result reduces it to $f(x) = \frac{3^n(2^i k_1) - 2^n}{2^n}$. Choosing sequence number i and letting n=i,, we get $f(x) = 3^n(k_1) - 1$, which is always an even number. since the product of odd numbers 3^n and k_1 is always odd and subtracting 1 from that product is always even. Note that for n less than i we would see an odd number in all previous sequence elements. Since only elements of A_5 can result in an even number when $\frac{3x+1}{2}$ is applied to it, and since all of that set obeys Collatz, we have a sequence element that obeys Collatz at index i-1. By Lemma 1, we have verified all of the set A_3 as obeying Collatz. Let A_e be the set of all even numbers in $\mathbb{N}$. When the Collatz function is applied to any $x_0 \in A_e$, the function causes the root to be divided by 2 until an odd number $x_b \in (A_3 \cup A_5)$ is obtained at some index of b. Since the element at index b is a number in the the sequence that obeys Collatz, by Lemma 1, so does x_0.

Since $\mathbb{N} = A_e \cup A_3 \cup A_5$, and since we have shown that all elements of those sets must obey Collatz, we have proven that Collatz is holds for all of $\mathbb{N}$

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