Proofs for Collatz Conjecture and Kaakuma Sequence

Dawit Geinamo
odawit@gmail.com, odawit@yahoo.com
Private Researcher

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Abstract

The objective of this study is to present precise proofs of the Collatz conjecture and introduce some interesting behavior on Kaakuma sequence. We propose a novel approach that tackles the Collatz conjecture using different techniques and angles. The Collatz Conjecture, proposed by Lothar Collatz in 1937, remains one of the most intriguing unsolved problems in mathematics. The conjecture posits that, for any positive integer, applying a series of operations will eventually lead to the number 1. Despite decades of rigorous investigation and countless computational verification, a complete proof has eluded mathematicians.

In this research endeavor, we embark on a comprehensive exploration of the Collatz Conjecture, aiming to shed light on its underlying principles and ultimately establish its validity. Our investigation begins by defining the Collatz function and investigating some behavior like. transformation, selective mapping, successive division, constant growth rate of inverse tree, and more. Using and analyzing the discovered properties of Collatz sequence we can show there are contradiction. In addition to this we investigate Qodaa ratio test that validates the reality of discovered behavior of Collatz sequence and works for infinite and distinct Kaakuma sequences.

Our investigation culminates in the formulation of a set of conjectures encompassing lemmas and postulates, which we rigorously prove using a combination of analytical reasoning, numerical evidence, and exhaustive case analysis. These results provide compelling evidence for the veracity of the Collatz Conjecture and contribute to our understanding of the underlying mathematical structure. This proof helps to change some researchers' views on unsolved problems and offers new perspectives on probability in infinite range. In this study, we uncover the dynamic nature of the Collatz sequence and provide a reflection and interpretation of the probabilistic proof of the Collatz Conjecture.

Keywords: Collatz Conjecture, 3x+1, number theory, mathematical proof, recursive sequences, computational analysis, modular arithmetic, Kaakuma Sequence, Qodaa ratio test, Stopping Time

1 Introduction

The Collatz Conjecture, also known as the 3n+1 Conjecture, Hailstone Problem, Kakutani's Conjecture, Ulam's Conjecture, Hasse's Algorithm, and the Syracuse Problem, is a long-standing and unsolved mathematical problem that has fascinated mathematicians for 87 years. It is one of the most dangerous unsolved problems in mathematics. The conjecture is named after German mathematician Lothar Collatz, who first proposed it in 1937.

Statement of the Conjecture

The Collatz Conjecture originally states an iterative sequence of natural numbers. Take a natural number n. If n is even, make it half. If n is odd, multiply it by 3 and add 1. Continue the process repeatedly, taking the result as the next input, and continue iterating. The conjecture states that regardless of the starting value, the sequence of numbers will eventually reach the value 1. For example:

$$14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5$$

$$\to 16 \to 8 \to 4 \to 2 \to 1$$

Historical Background and Significance

The Collatz Conjecture has captured the minds of mathematicians for almost a century. Many have attempted to prove or disprove it, employing various techniques and approaches. Despite its apparent simplicity, the conjecture has resisted all attempts at a definitive solution. The search for a solution to the Collatz Conjecture continues, driven by the allure of a seemingly simple problem harboring immense complexity. It serves as a reminder that even in the vast realm of mathematics, profound mysteries still await discovery.

Even though the Collatz Conjecture is simple to express and understand, it has tantalized scientists for around a century. Mathematicians have extensively tested the conjecture using computers for billions of billions of values, and it holds true for all tested cases. The Collatz Conjecture has fascinated mathematicians due to its apparent simplicity combined with its elusiveness.

Many attempts have been made to prove or disprove the conjecture, involving various mathematical techniques and concepts. However, the conjecture remains one of the most enduring unsolved problems in mathematics.

Heuristic Argument

A heuristic argument, sometimes stated as a probabilistic approach, attempts to show that the conjecture is true for infinitely diverging cases, not for non-trivial cycles, especially if the number of iterations is small to make a cycle. The probabilistic approach concerns how often each case will happen in mean to get lower or upper values of the starting number after a number of iterations. The ratio is $3/4$ and $n \to 3n/4$. This forms a basic study in research, working with varied examples.

Improved Results and Further Research

Almost all initial values n on which we perform our Collatz function T conclusively iterate to a value that is less than n. Studies indicate that 99.99% of starting values iterate to a value less than the starting value. Allouche and Korec have improved this result by proving that for an initial value n, it iterates to a value less than $n^{0.869}$ and more improved to a value less than $n^{0.7925}$, respectively. Terras's paper "A Stopping Time Problem on the Positive Integers" (Terras, 1976) provides initial derivation.

Allouche proves that almost all values iterate to a value less than $n^{0.869}$ and states that not just asymptotic behavior is required to determine the periodicity of the function, with periodicity referring to repeating points and intervals between them. The ideas used in Allouche's paper build on those used by Terras in his original proof and are continued by Ivan Korec (Korec, 1994).

Tao's contribution to the Collatz Conjecture (Tao, 2019) represents a significant breakthrough. His main result, "Collatz Orbits Attain Almost Bounded Values," states that for any function $f(n)$ such that when n tends to infinity, $f(n)$ also tends to positive infinity, the minimum term within a given Collatz orbit of n will be less than $f(n)$ for almost all values of n.

Kaakuma Sequence

The Kaakuma sequence is a general form of the Collatz sequence:

$$f(n) = \begin{cases} \frac{k_1 n c_1}{b_1} & \text{case } 1 \\ \frac{k_2 n c_2}{b_2} & \text{case } 2 \\ \frac{k_3 n c_3}{b_3} & \text{case } 3 \\ \vdots & \\ \frac{k_i n c_i}{b_i} & \text{case } i \end{cases}$$

All integers must be included in cases, no integer should be expressed in two cases, and a case should never be only self-cycled or semi-cycled except one cycle.

2 Expressions of Collatz sequence

The Collatz conjecture can be represented in different ways while retaining the same meaning. Below are various notations used to describe the conjecture.

a) General Notation

$$n_{i+1} = \begin{cases} 3n_i + 1 & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases}$$

Here, n_i is any number that begins an orbit and eventually reaches 1 by iterating rule.

b) Function Notation

$$f(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

In this notation, the result is used as the next value for iteration until the value reaches 1.

c) Simplified Notation

$$n = \begin{cases} 3n + 1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

This notation is often used in coding assignments. The right side of the equation is the input, and the left side is the output. The iteration continues using the output as the next input until reaching 1.

d) Shorter Form

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

e) Modular Form

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

This notation expresses the conditions of iteration in modular form.

f) Inverse of the Collatz Conjecture

The inverse of the Collatz conjecture states that if you start from 1 as a root of a tree, and for each number, you double it in all cases and divide a number minus one by three when it is possible to get a positive integer, then all natural numbers are traced in the tree map. This implies that no natural number is left out of the reverse tree map.

$$f(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ 2n & \forall n \ (n \in \mathbb{N}) \end{cases}$$

Table 1: Tabular form of Inverse Tree Map

$n \equiv 4 \pmod 6$	$f(n) = \frac{n-1}{3}$	$f(n) = 2n$
4	1	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024
16	5	10, 20, 40, 80, 160, 320, 640, 1280
10	3	6, 12, 24, 48, 96, 192, 384, 768, 1536
40	13	26, 52, 104, 208, 416, 832, 1664
52	17	34, 68, 136, 272, 544, 1088
34	11	22, 44, 88, 176, 352, 704, 1408
22	7	14, 28, 56, 112, 224, 448, 896, 1792
28	9	18, 24, 48, 96, 192, 384, 768, 1536
64	21	42, 84, 164, 328, 656, 1312
88	29	58, 116, 232, 464, 928, 1856
58	19	38, 76, 152, 304, 608, 1216
76	25	50, 100, 200, 400, 800, 1600
112	39	78, 156, 312, 624, 1248

In this tabular form of the inverse tree of the Collatz function, the nodes make new branches from values in the form $3k + 1$ from existing nodes.

3 Behavior of the Collatz Sequence

Before proceeding with the proof of the Collatz conjecture, it is essential to understand some basic behaviors of the Collatz sequence.

3.1 Transformation

3.1.1 Translation

Translation is a transformation that shifts each value in the orbit by a fixed distance forward or backward. For example:

$$\text{Original sequence: } 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$$

$$\text{Shifted by two forward: } 9, 24, 13, 36, 19, 54, 28, 15, 42, 22, 12, 5, 18, 10, 6, 4, 3$$

$$\text{Shifted by three backward: } 4, 19, 8, 31, 14, 49, 23, 10, 37, 17, 7, 2, 13, 5, 1$$

The function $f(n)$ and its translated version $g(n)$ can be expressed as:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$g(n) = f(n) + 2 = \begin{cases} \frac{3n-3}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Similarly,

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

$$g(n) = f(n) - 3 = \begin{cases} \frac{3n+7}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

*Translation Formula During translation of the sequence, only the constant terms are changed . The transformation can be expressed as:

$$f(c) = c - l(k - c)$$

If a conditional equation is $\frac{kn+c}{d}$ and it is translated by length l, then the translated equation becomes:

$$\frac{kn + c - l(k - d)}{d}$$

This formula is applied in all cases and is used with its sign or direction.

*Lemma 1 The next term n after shifting by translating length l is:

$$\frac{kn+c}{d}+l=\frac{kn+c+dl}{d}$$

Using the direct formula:

$$\frac{k(n+l)+c-l(k-d)}{d}=\frac{kn+c+dl}{d}$$

Proof can be carried out by induction.

*Short Form Translation For a short form of the Collatz sequence translated forward by 1:

$$f(n)=\begin{cases}\frac{3n+1}{2} & \text{if } n\equiv 1 \pmod 2\\ \frac{n}{2} & \text{if } n\equiv 0 \pmod 2\end{cases}$$

$$g(n)=f(n)+1=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n+1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

This form and its inverse is used for its simplicity in this study and .

3.1.2 Reflection on the Y-Axis

A reflection of the Collatz orbit on the y-axis involves multiplying constant terms by -1 and starting the sequence with the reflected value:

$$-1\times\frac{kn+c}{d}\longleftrightarrow\frac{kn-c}{d}$$

For the functions:

$$f(n)=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n+1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

$$g(n)=-f(n)=f(-n)=\begin{cases}\frac{3n}{2} & \text{if } n\equiv 0 \pmod 2\\ \frac{n-1}{2} & \text{if } n\equiv 1 \pmod 2\end{cases}$$

Example sequence for negative integers:

$$-8,-12,-18,-27,-14,-21,-11,-6,-9,-5,-3,-2$$

This converges to the $-2,-3$ cycle.

3.1.3 Scaling Up Mapping

Scaling involves multiplying the sequence by a fixed value s. This is done by multiplying the constant terms by the scaling natural number:

$$s \times \frac{kn+c}{d} \longleftrightarrow \frac{kn+sc}{d}$$

When the Collatz orbit is scaled up by s, e.g., multiplying by 5:

$$8, 12, 18, 27, 14, 21, 11, 6, 9, 5, 3, 2$$

multiplied by 5 yields:

$$40, 60, 90, 135, 70, 105, 55, 30, 45, 25, 15, 10$$

For the function:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = 5 \times f(n) = f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+5}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The scaled map of the Collatz sequence by a number different from a power of 3 has two or more cycles:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+3^i}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The trajectory converges to 2×3^i or $(2\times 3^i, 3^{i+1})$ cycle for all positive integers. For instance:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+27}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

This converges to 54 or $(54, 81)$ cycle.

3.2 Selective Mapping of Collatz Sequence

In selective mapping, only selected parts of the Collatz sequence or nearby nodes are mapped to a new sequence. Below are examples of selective mappings for different congruence conditions.

When $f(n) \equiv 0 \pmod 3$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{3} \quad \text{if} \quad f(n) \equiv 0 \pmod 3 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

It converges to 1 for all natural numbers.
28, 42, 63, 32, 48, 72, 108, 162, 243, 122, 183, 92, 138, 207, 104, 156, 234, 351, 176, 264, 396, 594, 891, 446, 669, 335, 168, 252, 378, 567, 284, 426, 639, 320, 480, 720, 1080, 1620, 2430, 3645, 1823, 912, 1368, 2052, 3078, 4617, 2309, 1155, 578, 867, 434, 651, 326, 489, 245, 123, 62, 93, 47
maps to:
14, 21, 16, 24, 36, 54, 81, 61, 46, 69, 52, 78, 117, 88, 132, 198, 297, 223, 56, 84, 126, 189,142, 213, 160, 240, 360, 540, 810, 1215, 304, 456, 684, 1026, 1539, 385, 289, 217, 163, 41, 31, 8

When $f(n) \equiv 0 \pmod 5$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{5} \quad \text{if} \quad f(n) \equiv 0 \pmod 5 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n & \text{if } n \equiv 3 \pmod 4 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \\ \frac{9n+7}{4} & \text{if } n \equiv 5 \pmod{32} \\ \frac{9n+11}{8} & \text{if } n \equiv 53 \pmod{64} \\ \frac{9n+67}{64} & \text{if } n \equiv 21 \pmod{64} \end{cases}$$

It converges to 1 for all natural numbers.

28, 42, 63, 32, 48, 72, 108, 162, 243, 122, 183, 92, 138, 207, 104, 156, 234, 351, 176, 264, 396, 594, 891, 446, 669, 335, 168, 252, 378, 567, 284, 426, 639, 320, 480, 720, 1080, 1620, 2430, 3645, 1823, 912, 1368, 2052, 3078, 4617, 2309, 1155, 578, 867, 434, 651, 326, 489, 245, 123, 62, 93, 47
maps to:
11, 33, 25, 19, 57, 43, 129, 97, 73, 55, 165, 373, 421, 949, 1069, 67, 201, 151, 453, 1021, 64, 96, 144, 216, 324, 486, 729, 547, 1641, 1231, 3693, 231, 693, 781, 49, 37

When $f(n) \equiv 0 \pmod 9$

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and

$$g(n) = \frac{f(n)}{9} \quad \text{if} \quad f(n) \equiv 0 \pmod 9 = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+1}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{9n+1}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod{32} \\ \frac{3n+5}{32} & \text{if } n \equiv 9 \pmod{32} \\ \frac{9n+7}{32} & \text{if } n \equiv 17 \pmod{32} \\ \frac{n+7}{64} & \text{if } n \equiv 57 \pmod{64} \\ \frac{9n+31}{128} & \text{if } n \equiv 25 \pmod{128} \\ \frac{3n+5}{16} & \text{if } n \equiv 89 \pmod{128} \end{cases}$$

It converges to 1 for all natural numbers.
28, 42, 63, 32, 48, 72, 108, 162, 243, 122, 183, 92, 138, 207, 104, 156, 234, 351, 176, 264, 396, 594, 891, 446, 669, 335, 168, 252, 378, 567, 284, 426, 639, 320, 480, 720, 1080, 1620, 2430, 3645, 1823, 912, 1368, 2052, 3078, 4617, 2309, 1155, 578, 867, 434, 651, 326, 489, 245, 123, 62, 93, 47
maps to:
7, 8, 12, 18, 27, 81, 23, 26, 39, 44, 66, 99, 297, 28, 42, 63, 71, 80, 120, 180, 270, 405, 152, 228, 342, 513, 385, 289, 217, 41, 4

When $f(n) \equiv 2 \pmod 3$:

$$f(n) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

and $\quad g(n) = \dfrac{f(n)-2}{3}$ if $f(n) \equiv 2 \pmod 3 = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$

3.2.1 Successive Division and Squeezing Stopping Time:

$$f(n) = \begin{cases} \frac{9n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{3n+3}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

Eg: $8, 18, 14, 11, 9, 3, 3$

$$f(n) = \begin{cases} \frac{27n}{8} & \text{if } n \equiv 0 \pmod 8 \\ \frac{9n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{9n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{3n+6}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{9n+9}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+7}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+9}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

Eg: $8, 27, 11, 5, 3, 2$

$$f(n) = \frac{3n + 3 \times 2^{i-1} - 3}{2^i} \text{ if } n = 2^i k + 2^{i-1} + 1$$

where i ranges from 1 to ∞ as we divide the last case into two infinitely.

3.3 Proportional Distribution of Powers of 3 or 2

When mapping the inverse tree of the Collatz trajectory, There are two occurrences of 3^{i-1} situated between two instances of 3^i on onward sequence $n = 2n$.

- There are only two $3^i k$ numbers between two $3^{i+1}k$ numbers.
- The maximum number of $3^{i+j}k$ numbers between two $3^i k$ numbers is only one, for i and j greater than 1.
- All $3k$ numbers are separated by only one $3k + 2$ number.

Example:
27, 53, 105, 209, 417, 833, 1665, 3329, 6657, 13313, 26625, 53249, 106497, 212993, 425985, 851969, 1703937, 3407873, 6815745

Lemma 2 For $3k$, $6k-1$, and $12k-3$, all pairs of $3k$ numbers are separated by one $3k+2$ number. From this, when we formulate sequences of $3k$ numbers:

$$f(n) = 4n - 3$$

$9k$, $36k - 3$, $144k - 15$, $576k - 63$, all pairs of $9k$ numbers are separated by two $3k$ numbers. From this, when we formulate sequences of $9k$ numbers:

$$f(n) = 64n - 63$$

By following the same principle, $3^i k$ can be formulated by $2^{2j}n - 3^i l$. If we start the sequence with $3^{i+1}k$, the sequence is:

$$\begin{aligned}
&3^{i+1}k,\\
&2^{2j}3^{i+1}k - 3^i l,\\
&2^{4j}3^{i+1}k - 2^{2j}3^i l - 3^i l,\\
&2^{6j}3^{i+1}k - 2^{4j}3^i l - 2^{2j}3^i l - 3^i l
\end{aligned}$$

where $j_1 = 1$ and $j_{i+1} = 3j_i$.

The fourth term is a factor of 3^{i+1} because j is even and $2^{4j} + 2^{2j} + 1$ is a factor of 3:

$$\begin{aligned}
2^{4j}n &\equiv 1 \pmod 3,\\
2^{2j}n &\equiv 1 \pmod 3,\\
1 &\equiv 1 \pmod 3
\end{aligned}$$

Adding them:

$$2^{4j} + 2^{2j} + 1 \equiv 0 \pmod 3$$

Thus:

$$2^{6j}3^{i+1}k - 2^{4j}3^{i}l - 2^{2j}3^{i}l - 3^{i}l = 3^{i}\left(3 \times 2^{6j} - (2^{4j}l + 2^{2j}l + 1)\right) = 3^{i+1}m$$

Therefore, the load of the tree or branches of the inverse of the Collatz function is proportional to their root or branch nodes. This behavior of the Collatz sequence maintains the proportionality of branch loads and prevents the occurrence of unexpected behavior in the sequence. This property is one of crucial property to decide collat conjecture.

The inverse of this statement is in the Collatz sequence might result in one branch having few or no high powers of 3, while another branch has many high powers of 3, with the same starting condition and height. This leads to significant variability in the growth rate of branches and no proportional growth rates among branches.

3.4 Constants

3.4.1 Nearly Constant Expansion Rate of Inverse Tree Map

The average growth rate of the Collatz inverse tree map is $\frac{1}{3}$.

$$f(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ 2n - 1 & \text{if } n \text{ is any natural number} \end{cases}$$

Let us start from 2 as the root of the tree and ignore recycling because the 2 and 3 cycling cases duplicate data. The main root of the tree is 2, $\{2\}$, $\{3\}$, $\{5\}$, $\{9\}$, $\{6, 17\}$, $\{4, 11, 33\}$, $\{7, 21, 22, 65\}$, ...

3.4.2 Expansion Rate Analysis

The expansion rate, on average, is $\frac{1}{3}$. For lists with more than 30 elements, it is expected that $\frac{1}{3}$ of the numbers are of the form $3k$, $\frac{1}{3}$ are of the form $3k+1$, and $\frac{1}{3}$ are of the form $3k+2$. Among these, $3k$ creates double nodes $6k-1$ and $2k$. That is why the expansion rate is $\frac{1}{3}$.

H, LC, TS is for Height, Leaf Count and Tree Size respectivily

H	LC	TS	Leafs
1	1	1	2
2	1	2	3
3	1	3	5
4	1	4	9
5	2	6	6, 17
6	3	9	4, 11, 33
7	4	13	7, 21, 22, 65
8	5	18	13, 14, 41, 43, 129
9	6	24	25, 27, 81, 85, 86, 257
10	8	32	49, 18, 53, 54, 161, 169, 171, 513
11	12	44	97, 12, 35, 105, 36, 107, 321, 337, 114, 341, 342, 1025
12	18	62	193, 8, 23, 69, 70, 209, 24, 71, 213, 214, 641, 673, 76, 227, 681, 228, 683, 2049
13	24	86	385, 15, 45, 46, 137, 139, 417, 16, 47, 141, 142, 425, 427, 1281, 1345, 151, 453, 454, 1361, 152, 455, 1365, 1366, 4097
14	31	117	769, 10, 29, 30, 89, 91, 273, 277, 278, 833, 31, 93, 94, 281, 283, 849, 853, 854, 2561, 2689, 301, 302, 905, 907, 2721, 303, 909, 910, 2729, 2731, 8193
15	39	156	1537, 19, 57, 20, 59, 177, 181, 182, 545, 553, 555, 1665, 61, 62, 185, 187, 561, 565, 566, 1697, 1705, 1707, 5121, 5377, 601, 603, 1809, 1813, 1814, 5441, 202, 605, 606, 1817, 1819, 5457, 5461, 5462, 16385
16	50	206	3073, 37, 38, 113, 39, 117, 118, 353, 361, 363, 1089, 1105, 370, 1109, 1110, 3329, 121, 123, 369, 373, 374, 1121, 1129, 1131, 3393, 3409, 1138, 3413, 3414, 10241, 10753, 1201, 402, 1205, 1206, 3617, 3625, 3627, 10881, 403, 1209, 404, 1211, 3633, 3637, 3638, 10913, 10921, 10923, 32769

Table 2: Tree growth data

The table above shows the leaf count in each step with new branches approaching a size $\frac{1}{3}$ of the previous leaf count.

H	L Count	T Size	Rate	H	L Count	T Size	Rate
1	1	1		30	2829	11301	33.317
2	1	2	0	31	3765	15066	33.085
3	1	3	0	32	5014	20080	33.1739
4	1	4	0	33	6682	26762	33.266
5	2	6	100	34	8902	35664	33.223
6	3	9	50	35	11878	47542	33.430
7	4	13	33.333	36	15844	63386	33.389
8	5	18	25	37	21122	84508	33.312
9	6	24	20	38	28150	112658	33.273
10	8	32	33.333	39	37536	150194	33.342
11	12	44	50	40	50067	200261	33.383
12	18	62	50	41	66763	267024	33.347
13	24	86	33.333	42	89009	356033	33.320
14	31	117	29.166	43	118631	474664	33.279
15	39	156	25.806	44	158171	632835	33.330
16	50	206	28.205	45	210939	843774	33.361
17	68	274	36	46	281334	1125108	33.372
18	91	365	33.823	47	375129	1500237	33.339
19	120	485	31.868	48	500106	2000343	33.315
20	159	644	32.5	49	666725	2667068	33.316
21	211	855	32.704	50	888947	3556015	33.330
22	282	1137	33.649	51	1185305	4741320	33.338
23	381	1518	35.106	52	1580518	6321838	33.342
24	505	2023	32.545	53	2107346	8429184	33.332
25	665	2688	31.683	54	2809845	11239029	33.335
26	885	3573	33.082	55	3746399	14985428	33.331
27	1187	4760	34.124	56	4995078	19980506	33.330
28	1590	6350	33.951	57	6660211	26640717	33.335
29	2122	8472	33.459				

Table 3: Growth Rate of Leaf Count with Heights

The table above shows the leaf count and tree size at each height with their corresponding rate of expansion. The average expansion rate remains close to $\frac{1}{3}$ as the tree grows.

3.4.3 Average Stopping Time

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Average stopping time of this sequence is 3.49269.

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Average stopping time of this sequence is 3.49269.

3.4.4 Ratio of Stopping Time

The ratio of stopping time to $\log_2(n)$ is bounded. Specifically, this ratio is less than 10 for large starting numbers (more than 8 digits). For such large numbers, the ratio is bounded and typically less than 6. For a starting number 2^p and stopping time t, the ratio ranges from 3.67 to 5.15.

p	187	188	189	190	191	192	193	194	195	196	197
t	693	690	753	753	753	749	994	994	994	994	747
t/p	3.71	3.67	3.98	3.96	3.94	3.90	5.15	5.12	5.10	5.07	3.79

Table 4: Ratio of stopping time t to $\log_2(n)$

3.4.5 Ratio of Iteration Time

The ratio of iteration time to reach 1 (T) to $\log_2(n)$ is bounded. Specifically, this ratio is less than 15 and tends to approximately 8.9 for most high-iteration cases involving large numbers.

p	115543	200000	200001	200002	200003	200004	200005
T	1001348	1728481	1728482	1728483	1728484	1728485	1728486
T/p	8.66645	8.64240	8.64236	8.64232	8.64229	8.64225	8.64221

Table 5: Ratio of iteration time T to $\log_2(n)$

3.5 Stopping Time Iteration Groups

When we group numbers by iteration, some numbers have the same number of stopping times and are grouped by $2^t k + c$. If the iterations of c's stopping time is t and $2^t > c$, then all numbers formed by $2^t k + c$ have stopping time t.

t	4	7	5	7	5	59	56	8	54	7	54	51	8	45	8
c	4	8	12	16	24	28	32	40	48	60	64	72	80	92	96

Table 6: Stopping Time Iteration Groups

For corresponding values t to c, $2^t k + c$ has the same stopping time of t. For example:

- The stopping time t of $2^5 k + 12$ is 5,
- The stopping time t of $2^7 k + 16$ is 7,
- The stopping time t of $2^5 k + 24$ is 5,
- The stopping time t of $2^5 9k + 28$ is 59.

Riho Terras (1976) showed that almost all initial values (more than 99.99%) eventually become a value less than n. This is 100 times the sum of the reciprocals of stopping times grouped: $100 \times \sum 1/2^t$.

3.6 Huge Iteration if There Exist Non-Trivial Cycle

If there is a non-trivial cycle, the number of iterations is very large, nearly equal to the starting number, and it is relatively easy to get large starting numbers. The sequence oscillates up and down while eventually returning to the starting number or the smallest number in the cycle.

The iteration sequence can be described as:

$$n,\ \frac{3n}{2},\ \frac{9n}{4},\ \frac{27n}{8},\ \frac{27n+8}{16},\ \frac{81n+24}{32},\ \frac{81n+56}{64},\ \frac{243n+168}{128},\ \ldots,\ \frac{3^u n + c}{2^t}$$

Where:

$$n = \frac{3^u n + c}{2^t - 3^u} = \frac{c}{2^t - 3^u}$$

Here, c is a partial geometric series with ratio $r = \frac{3}{4}$:

$$g_1 = 2^2 \cdot 3^{(u-2)},\ 2^3 \cdot 3^{(u-3)},\ 2^4 \cdot 3^{(u-4)},\ \ldots,\ 2^i \cdot 3^{(u-i)}$$

For c, it satisfies:

$$3^u < c < 3^{(u+1)} \quad \text{or} \quad 2^{(t+1)} < c < 2^{(t+3)}$$

The maximum value of c is given by:

$$n < \frac{3^{(u+1)}}{2^t - 3^u} \quad \text{or} \quad n_{\max} = \frac{3^{(u+1)}}{2^t - 3^u}$$

$$t = \lceil u \cdot \log_2 3 \rceil$$

and the difference $\lceil u \cdot \log_2 3 \rceil - u \cdot \log_2 3$ is very small to get big starting number n, too $2^t - 3^u$ small. Therefore, starting numbers n with high powers of 2 are less likely occure non-trivial cycles.

Here is the data for various values:

p	$\log_3(c)$	$t \cdot \log_2 3$	$\log_2(c)$	t	$\log_2(n_t)$	u
4004	3352.8402	3351.4988	5314.1259	5312	4003.2093	3351
4124	3266.5908	3265.0615	5177.424	5175	4123.9026	3265
4244	3294.6855	3293.4533	5221.953	5220	4243.2815	3293
4364	3548.7111	3547.0871	5624.574	5622	4363.862	3547
4484	3574.6895	3573.5861	5665.7488	5664	4483.071	3573
4604	3579.7600	3578.0026	5673.7854	5671	4603.9958	3578
4724	3736.5523	3735.1041	5922.2952	5920	4723.8349	3735
4844	3821.6234	3820.2797	6057.1297	6055	4843.5568	3820
4964	4108.7147	4107.3527	6512.1587	6510	4963.441	4107
5084	4093.6378	4092.2104	6488.2624	6486	5083.6665	4092
5204	4235.6435	4234.1696	6713.3361	6711	5203.7312	4234
5324	4388.6924	4387.4855	6955.9129	6954	5323.2305	4387
5444	4417.8300	4416.5083	7002.0948	7000	5443.1944	4416
5564	4437.6055	4436.0671	7033.4384	7031	5563.8937	4436
5684	4510.6267	4509.2549	7149.1742	7147	5683.5959	4509
5804	4722.8123	4721.2473	7485.4804	7483	5803.6080	4721
5924	4787.6952	4786.2331	7588.3173	7586	5923.6305	4786

Table 7: Analysis of Stopping Time and Starting Number

If there exist non-trivial cycle

When $n = 2^p$ and $2^t < c < 2^{(t+1)}$, if 2^p is very high, a non-trivial loop is less likely to occur because $n = \frac{c}{2^t - 3^u} < 2$. Therefore, the most expected starting number n for a non-trivial cycle of $c(n)$ is $8k + 4$ or a number with only a few powers of 2 greater than one.

When evaluating the starting number n with possible stopping time t, the formula used is:

$$n = \frac{c}{2^t - 3^u}$$

To check if we can get big starting number with given stopping time.

- For the first row:

$$n = \frac{3^{3352.8402}}{3^{3351.4988} - 3^{3351}} = \frac{3^{1.8402}}{3^{0.4988} - 3^0} \approx 10.347 \approx 10$$

  This value is already known as $c(n)$.

- For the second row:

$$n = \frac{3^{3266.5908}}{3^{3265.0615} - 3^{3265}} = \frac{3^{1.5908}}{3^{0.0615} - 3^0} \approx 82.135 \approx 82$$

  This value is also already known as $c(n)$.

- For the sixth row:

$$n = \frac{3^{3579.7600}}{3^{3578.0026} - 3^{3578}} = \frac{3^{1.7600}}{3^{0.0026} - 3^0} \approx 2417.103 \approx 2417$$

  This value is known as $c(n)$.

This shows that when $t \cdot \log_2 3 - u$ approaches 0, it is crucial to check the results.

From computer searches, the maximum value of n is less than 3^t. For a 20-digit number, if it is not a known $c(n)$ and is looped, it must have at least 19 digits in the number of iterations. Analysis suggests that a number of height $c(h)$ is expected to be around 45 digits.

3.7 Bigger Stopping Time Relatively

The shortest stopping time is known to be 1 for all odd numbers n, which follows the sequence $\frac{n+1}{2}$. Additionally, the stopping time for numbers of the form $4k+2$ is 2, and for numbers of the form $6k+3$ or $3k+2$, the stopping time is also determined by specific rules. However, the challenge is to find numbers with very large stopping times.

Finding Huge Stopping Times

To find large stopping times, we start by considering powers of 2, 2^k are known to have high stopping times. If a sequence reaches 2^k before the stopping time, it can achieve relatively higher stopping times. For instance, sequences that follow the pattern

$$4k \to 4k \to 4k \to 2k \to 2^k$$

demonstrate this phenomenon, as they lead to higher stopping times. This approach highlights that large iterations can be achieved when the sequence reaches 2^k before the stopping time.

Bound on Stopping Time Ratio

Despite these higher stopping times, the ratio of stopping time t to $\log_2(n)$ remains bounded. Specifically, this ratio is observed to be around 3.85 for such sequences, indicating that while stopping times can be large, their growth is relatively controlled when compared to the logarithm of the starting number.

4 Proofs

4.1 Proof1 Contradiction in Tree Size Density

As we have established in Section 3.4 regarding the balance of branches in the inverse tree map of the Collatz sequence, and observed in Section 3.5.1 regarding the constant growth of this inverse tree map, we can now determine the density of non-Collatz sequences relative to Collatz sequences.

Consider 2^{80} as the first non-Collatz number. We approach the problem using two methods to compare the tree size of the inverse tree map for Collatz and non-Collatz sequences:

1. **First Method:** Grow the tree until the largest leaf of the Collatz inverse tree exceeds 2^{80}, which occurs at height 81, with a leaf count greater than 6.6×10^9. At this height, the largest leaf is $2^{80} + 1$. This minimum leaf count allows comparison with non-Collatz sequences because 2^{80} - 6.6×10^9 nodes are traced back, making the tree denser, whereas 2^{80} will never trace a number smaller than itself. If $3^{50} < 2^{80}$ is a leaf at height 81, it will generate many more leaves less than 2^{80} at different heights. Therefore, the density of the Collatz inverse tree map is much greater than 6.6×10^9 times the density of the non-Collatz inverse tree.

2. **Second Method:** Count all leaves less than 2^{80} exactly using computer code, which may take months on a PC. Nonetheless, the minimum leaf count of 6.6×10^9 is significant enough to demonstrate a contradiction.

This contradiction is grounded in the selective mapping discussed in property 3.2. Let

$$f(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3, \\ 2n - 1 & \forall n \ (n \in \mathbb{N}), \end{cases}$$

representing the inverse tree map of the Collatz sequence. If $2p$ is the first non-Collatz number and $g(n) = f(n) \ / \ p$ when $f(n) \equiv 0 \pmod p$, then the first non-Collatz number maps to 2. The numbers 2, 3, 4, 6, and 9 are known nodes of the non-Collatz inverse tree from the beginning, which makes it denser than the Collatz inverse tree with only the known root 1.

If $2p$ is a non-Collatz number, then $4p$ is also a node of the non-Collatz sequence. Therefore, numbers 2, 3, 4, 6, and 9 are part of the non-Collatz sequence after mapping. Beyond this cycling or infinitely diverging makes denser non-Collatz inverse tree than collatz Inverse tree after mapping.

Thus, there is a significant contradiction in the density relationship between the inverse trees of Collatz and non-Collatz sequences. This suggests that non-Collatz sequences do not exist, thereby supporting the Collatz conjecture.

Note: To apply the tree balance test, numbers less than the square of the product of the coefficients of all conditions must be tested. Significant contradictions arise after considering the product of coefficients, with growth rates becoming constant. After the square of this product for large numbers all expected constants become stable.

4.2 Proof2 Qodaa Ratio Test

The Qodaa Ratio Test is a method of analyzing the product of coefficients of cases with their occurrences as power of a Kaakuma sequence. Kaakuma sequence is a sequence of integers that fluctuating up and down based on conditions and it is equated with two or more well defined conditions. The Kaakuma sequence is a broad generalization of the Collatz sequence. The Qodaa Ratio Test helps in determining the exact limit coefficients where diverging occurs by examining the ratio of products of numerators to denominators with their occurrences.

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{3n-2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{n}{4} & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

Case1 2k+1 it is half of natural numbers, it generates only one-fourth of natural numbers of case2 and one-fourth of natural numbers of case3 with ratio case2:case3=1/4:1/4=1:1.
Case2 4k+2 it is one-fourth of natural numbers, it generates half of natural numbers of case1, one-fourth of natural numbers of case2 and one-fourth of natural numbers of case3 with ratio case1:case2:case3=1/2:14:1/4=2:1:1 based on their fractions of natural numbers and Case3 4k it is one-fourth of natural numbers, it generates in the same with case2 When we calculate them by in-out rule they may have different occurrences amount of cases

relatively. The occurrences amount of each case is used as the power of cases in product of coefficients.
Before starting we need to realize some points on Qodaa ratio as much as Qodaa Ratio Test is efficient and simple to apply.

- If cases do not have proportional chances of generating other cases, then the tree size of branches on the inverse tree map of the Kaakuma sequence is not applicable and nearly constant growth of leaves is not valid. Proportional cases generation validates tree size balance and vice versa.
- If cases do not have proportional chances of generating cases, then the generating amount must be negligible to avoid overload of tree size.
- The occurrences of cases, number of iterations, and occurrence of values are not random, even if they cannot be precisely determined. It is possible to infer them from behaviors discussed in Sections 3.3 (Successive Case Division) and 3.6 (Stopping Time Iteration Group).
- Even if occurrences are probabilistic, values like 3/4 must be interpreted and defined carefully, particularly as probabilistic value approach zero.
- if we force to vary natural law of generating of cases proportionally it is impossible to set rule when altered by successive partition or selective mapping .

$$f(n) = \begin{cases} \frac{k_1 n + c_1}{b_1} & \text{Case 1} \\ \frac{k_2 n + c_2}{b_2} & \text{Case 2} \\ \frac{k_3 n + c_3}{b_3} & \text{Case 3} \\ \vdots & \vdots \\ \frac{k_i n + c_i}{b_i} & \text{Case } i \end{cases}$$

Qodaa Ratio Test states that if

$$\prod k_i^{p_i} < \prod b_i^{p_i},$$

then the sequence either never diverges to infinity or does not have a large non-trivial cycle based on $\prod k_i^{p_i}$. When applying the in-out rule, these cases may have different occurrences. The occurrences of each case are used as the power of cases in the product of coefficients. Kaakuma sequences have many categories. Among them, we can check simple, complex, and complicated Kaakuma sequences only for positive integers.

4.2.1 Simple Kaakuma Sequence

In a simple Kaakuma sequence, each case generates all cases, and we can simply take the ratio of the cases' fractions of natural numbers to determine the occurrences of each case relatively. This will be consistent with the rule of in and out.

Example 1: Base Two

$$f(n) = \begin{cases} \frac{kn+c}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Case 1: $n \equiv 1 \pmod 2$ is half of the natural numbers, and Case 2: $n \equiv 0 \pmod 2$ is also half of the natural numbers.

The ratio of Case 1 to Case 2 is $1/2 : 1/2 = 1 : 1$. From the Qodaa ratio test rule:

$$\left(\frac{k}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 < 1 \implies \frac{k}{4} < 1 \implies k < 4$$

The sequence $f(n)$ with $k = 3$:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio 3/4.

Example 2: Base Three

$$f(n) = \begin{cases} \frac{kn+c}{3} & \text{if } n \equiv 2 \pmod 3 \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{3} & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

The ratio is $1/3 : 1/3 : 1/3 = 1 : 1 : 1$. by using Qodaa ratio rule:

$$\left(\frac{k}{3}\right)^1 \times \left(\frac{1}{3}\right)^1 \times \left(\frac{1}{3}\right)^1 < 1 \implies k/27 < 1 \implies k < 27$$

With $k = 26$:

$$f(n) = \begin{cases} \frac{26n-25}{3} & \text{if } n \equiv 2 \pmod 3 \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{3} & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $26/27$.

Example 3: Base Four

$$f(n) = \begin{cases} \frac{255n-261}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n}{4} & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $255/256 = 0.996$.

Compare this with the original Collatz sequence after the first successive division:

$$f(n) = \begin{cases} \frac{9n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3n+2}{4} & \text{if } n \equiv 2 \pmod 4 \\ \frac{3n+3}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

converges to 2 for all $n \in \mathbb{N}$ with a Qodaa ratio of $81/256 = 0.3045$.

Example 4: Base Eight

$$f(n) = \begin{cases} \frac{16777215n-116440489}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+2}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n}{8} & \text{if } n \equiv 0 \pmod 8 \end{cases}$$

converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $16777215/16777216 = 0.99999994$.

Compare this with the original Collatz sequence after the second division:

$$f(n) = \begin{cases} \frac{27n}{8} & \text{if } n \equiv 0 \pmod 8 \\ \frac{9n+4}{8} & \text{if } n \equiv 4 \pmod 8 \\ \frac{9n+6}{8} & \text{if } n \equiv 2 \pmod 8 \\ \frac{3n+6}{8} & \text{if } n \equiv 6 \pmod 8 \\ \frac{9n+9}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{3n+7}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{3n+9}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

converges to 2 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.0317.

Example 5: Base Five

$$f(n) = \begin{cases} \frac{3124n-3131}{5} & \text{if } n \equiv 4 \pmod 5 \\ \frac{n+2}{5} & \text{if } n \equiv 3 \pmod 5 \\ \frac{n+3}{5} & \text{if } n \equiv 2 \pmod 5 \\ \frac{n+4}{5} & \text{if } n \equiv 1 \pmod 5 \\ \frac{n}{5} & \text{if } n \equiv 0 \pmod 5 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.99968.

Example 6: Base Six

$$f(n) = \begin{cases} \frac{46655n-46657}{6} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n+2}{6} & \text{if } n \equiv 4 \pmod 6 \\ \frac{n+3}{6} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n+4}{6} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n+5}{6} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{6} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999978.

Example 7: Base Seven

$$f(n) = \begin{cases} \frac{823542n-4200008}{7} & \text{if } n \equiv 6 \pmod 7 \\ \frac{n+2}{7} & \text{if } n \equiv 5 \pmod 7 \\ \frac{n+3}{7} & \text{if } n \equiv 4 \pmod 7 \\ \frac{n+4}{7} & \text{if } n \equiv 3 \pmod 7 \\ \frac{n+5}{7} & \text{if } n \equiv 2 \pmod 7 \\ \frac{n+6}{7} & \text{if } n \equiv 1 \pmod 7 \\ \frac{n}{7} & \text{if } n \equiv 0 \pmod 7 \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999998.

Example 8: Base Two with Sub-Cases

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

The cases share a ratio of $1/2 : 1/4 : 1/4 = 2 : 1 : 1$. The Qodaa ratio is:

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{k}{4}\right)^1 \times \left(\frac{1}{4}\right)^1 = \frac{k}{64}$$

We can use the Qodaa Ratio Test to determine the values of k. For the condition $k/64 < 1$, we have $1 < k < 64$ for positive integer values of k.

Produced	**Generates**			**After solved**			sum	Simplified
	A	B	C	A	B	C		
a	2a	2b	2c	4c	2c	2c	8c	2
b	a	b	c	2c	c	c	4c	1
c	a	b	c	2c	c	c	4c	1

Tabular Analysis of occurrences using in = out rule

When we equate the generating and generated values of each case using the in-out rule:

$$a = b + c \quad 3b = a + c \quad 3c = a + b \quad b = c \quad a = 2c$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{63n-59}{4} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $\frac{63}{64}$.

Example 9:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

The ratio of cases is 1/2 : 1/4 : 1/8 : 1/8, which simplifies to 4 : 2 : 1 : 1. The occurrences ratio yields:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{16384}$$

For positive integer values, $1 < k < 16384$.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{16383n-81907}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 0.999939.

when we set k in case2:

If we set k in line 2, the product of coefficient values differs due to the difference in power:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

To determine the limit of k using Qodaa ratio rule:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{k}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^2}{16384} \implies 1 < k < 128$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{127n-369}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

This sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of 127/128.

when we set k in case1:

$$f(n) = \begin{cases} \frac{kn+c}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod{8} \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

Using Qodaa ratio rule

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^4}{16384} \implies 1 < k < \sqrt{128}$$

$$f(n) = \begin{cases} \frac{11n-2}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

The sequence converges to 1 for all $n \in \mathbb{N}$ with a Qodaa ratio of $11/\sqrt{128}$.

Example 10

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n-3}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 7 \pmod 8 \end{cases}$$

With ratio $1/2 : 1/8 : 1/8 : 1/8 : 1/8 = 4 : 1 : 1 : 1 : 1$:

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{65536} \implies 1 < k < 65536$$

when we substitute k

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n-3}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+1}{8} & \text{if } n \equiv 7 \pmod 8 \\ \frac{65535n-327667}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Converges to 0 for all $n \in \mathbb{N}$, with $QR = 65535/65536$.

When shifting the coefficient in the first line:

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 \times \left(\frac{1}{8}\right)^1 < 1 \implies 1 < k < 16$$

$$f(n) = \begin{cases} \frac{15n-28}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+1}{8} & \text{if } n \equiv 7 \pmod 8 \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = 15/16$.

Example 11

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{kn+c}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

With ratio $1/2 : 1/4 : 1/8 : 1/16 : k/16 = 8 : 4 : 2 : 1 : 1$:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{30}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{2^{30}n-n-13\times 2^{30}+45}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = \frac{2^{30}-1}{2^{30}}$.

When shifting k in line 3:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{k}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{15}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{32767n-32751}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

When shifting k in line 2:

$$\left(\frac{1}{2}\right)^8 \times \left(\frac{k}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \implies 1 < k < 2^{15/2}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{181n-535}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

When shifting k in line 1:

$$\left(\frac{k}{2}\right)^8 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{1}{8}\right)^2 \times \left(\frac{1}{16}\right)^1 \times \left(\frac{k}{16}\right)^1 < 1 \Rightarrow 1 < k < 2^{15/4}$$

$$f(n) = \begin{cases} \frac{13n-4}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+7}{8} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+11}{16} & \text{if } n \equiv 5 \pmod{16} \\ \frac{n+3}{16} & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$.

4.2.2 Complex Kaakuma Sequence

In a complex Kaakuma sequence, at least one case is never generated by one or more cases. To analyze limit of converging values complex Kaakuma sequence, we use a tabular format to get the relative occurrence of each case.

Example 12

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

(original Collatz sequence)

We organize and represent each line of conditions or cases with capital letters A, B, C to show producing amounts and small letters a, b, c to show produced amounts with their order.

Produced	Produces		Solved in Terms of b		Sum	Simplified
	A	B	A	B		
a		b		b	b	1
b	a	b	b	b	$2b$	2

$a = b$, $QR = 3^3 \times \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

Example 13

$$f(n) = \begin{cases} 3n+1 & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, with $QR = \frac{27}{64}$.

Generated	Generates			Solved in Terms of c			Sum	Simplified
	A	B	C	A	B	C		
a			$2c$			$2c$	$2c$	1
b	a	b	c	c	$2c$	c	$4c$	2
c	a	b	c	c	$2c$	c	$4c$	2

$2a = 2c$, $b = a + c$, $3c = a + b$, $a = c$, $b = 2c$, $QR = 3^1 \times \left(\frac{3}{2}\right)^2 \times \left(\frac{1}{4}\right)^2 = \frac{27}{64}$

Example 14

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{kn+c}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 4 \end{cases}$$

When we use the generating and generated of each case, in=out rule.

Produced	Produces			Solved in Terms of a			Sum	Simplified
	A	B	C	A	B	C		
a	$2a$	$2b$		$2a$	$2a$		$4a$	1
b	a	b	c	a	a	$2a$	$4a$	1
c	a	b	c	a	a	$2a$	$4a$	1

$2a = 2b$, $3b = a + c$, $c = a + b \to a = b$, $c = 2a$

$$\left(\frac{1}{2}\right)^1 \times \left(\frac{k}{4}\right)^1 \times \left(\frac{1}{2}\right)^1 = \frac{k^1}{2^4} \to 1 < k^1 < 2^4 \to 1 < k < 2^4$$

Example 15

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Produced	Produces				Solved in Terms of d				Sum	Simplified
	A	B	C	D	A	B	C	D		
a	$4a$	$4b$		$4d$	$12d$	$8d$		$4d$	$24d$	3
b	$2a$	$2b$	$2c$	$2d$	$6d$	$4d$	$4d$	$2d$	$16d$	2
c	a	b	c	d	$3d$	$2d$	$2d$	d	$8d$	1
d	a	b	c	d	$3d$	$2d$	$2d$	d	$8d$	1

$a = b + d$, $3b = a + c + d$, $3c = a + b + d$, $7d = a + b + c$, so $c = 2d$, $b = 2d$, $a = 3d$

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k^1}{2^{12}} \to 1 < k^1 < 2^{12} \to 1 < k < 2^{12}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{4095n-20459}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= \frac{4095}{4096}$

When we shift k in line 2:

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{k}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^2}{2^{12}} \to 1 < k^2 < 2^{12} \to 1 < k < 2^6$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{63n-181}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= \frac{63}{64}$

When we shift k in line 1:

$$\left(\frac{k}{2}\right)^3 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{1}{8}\right)^1 = \frac{k^3}{2^{12}} \to 1 < k^3 < 2^{12} \to 1 < k < 2^4$$

$$f(n) = \begin{cases} \frac{15n-4}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 8 \\ \frac{n+3}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

converges to 1 $\forall n : n \in \mathbb{N}$, QR $= 15/16$

Example 16

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 8 \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod 8 \end{cases}$$

Produced	Produces				Solved in Terms of d				Sum	Simplified
	A	B	C	D	A	B	C	D		
a	$4a$			$4d$	$4d$			$4d$	$8d$	1
b	$2a$	$2b$		$2d$	$2d$	$4d$		$2d$	$8d$	1
c	a	b	c	d	d	$2d$	$4d$	d	$8d$	1
d	a	b	c	d	d	$2d$	$4d$	d	$8d$	1

$$a = d, \quad b = a + d, \quad b = 2d, \quad c = a + b + d, \quad c = 2b, \quad 7d = a + b + c$$

From these equations, we find:

$$c = 4d, \quad b = 2d, \quad a = d$$

$$\left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{k}{8}\right)^1 = \frac{k}{2^3}$$

$$\frac{k}{2^3} = \frac{k}{8} = \frac{k^1}{2^6}$$

$$1 < k^1 < 2^6 \to 1 < k < 2^6$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod{8} \\ \frac{55n+197}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

The function $f(n)$ converges to 1 for all $n \in \mathbb{N}$, and $QR = \frac{55}{64}$.

Example 17

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod{8} \\ \frac{n-5}{2} & \text{if } n \equiv 7 \pmod{8} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod{8} \\ \frac{kn+c}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

Produced	Produces					Solved in Terms of e					Sum	Simplified
	A	B	C	D	E	A	B	C	D	E		
a	$4a$				$4e$	$4e$				$4e$	$8e$	4
b	a				e	e				e	$2e$	1
c	a				e	e				e	$2e$	1
d	a	b	c	d	e	e	e	e	$4e$	e	$8e$	4
e	a	b	c	d	e	e	e	e	$4e$	e	$8e$	4

$$a = e, \quad 2b = a + e, \quad 2c = a + e, \quad d = a + b + c + e, \quad 7e = a + b + c + d$$

$$a = b = c = e, \quad d = 4e$$

$$\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{k}{8}\right)^4 = \frac{k^4}{2^{22}}$$

$$1 < k^4 < 2^{22} \to 1 < k < 2^{5.5}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod{8} \\ \frac{n-5}{2} & \text{if } n \equiv 7 \pmod{8} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod{8} \\ \frac{45n-33}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, $QR = 45/32\sqrt{2}$.

Example 18

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{kn+c}{2} & \text{if } n \equiv 1 \pmod{4} \end{cases}$$

Generated	Generates			Solved in Terms of a			Sum	Simplified
	A	B	C	A	B	C		
a	$2a$	$2b$	0	$2a$	$2a$	0	$4a$	2
b	a	b	c	a	a	$2a$	$4a$	2
c	a	b	0	a	a	0	$2a$	1

$$a = b, \quad 3b = a + c, \quad c = a + b, \quad c = 2a = 2b$$

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{k}{2}\right)^1 < 1 \implies 1 < k < 2^7$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \\ \frac{126n-120}{2} & \text{if } n \equiv 1 \pmod{4} \end{cases}$$

Converges to 1 for all $n \in \mathbb{N}$, $QR = \frac{63}{64}$.

Example 19

$$f(n) = \begin{cases} \frac{3n+3\cdot 2^{i-1}-3}{2^i} & \text{if } n = 2^i k + 2^{i-1} + 1 \text{ for } i \geq 1 \end{cases}$$

where i ranges from 1 to ∞.

Converges to 3 for all $n \in \mathbb{N}$ with $n > 1$ and QR $\to 0$.

4.2.3 Complicated Kaakuma Sequence

Equations with partially generating cases are impossible to apply the Qodaa ratio test directly. This highlights the elegance of the Qodaa ratio test and its insightful application to any well-stated Kaakuma sequence.

If it is not done with care and attention, it will be full of subtle errors.

Example 20

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

Case3 generates 6k, 6k+1, 6k+3 and 6k+4 that is 2/3 of case1, case2 and 1/2 of case3. The occurrences of a case also partially differ, to avoid subtle errors we have to dismantle all cases.

$$f(n) = \begin{cases} \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

ed	Produces						Solved in Terms of b						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a	a	b	c				$3b$	b	$2b$				$6b$	3
b				d						$2b$			$2b$	1
c			c		e	f			$2b$		$2b$	$2b$	$6b$	3
d	a	b					$3b$	b					$4b$	2
e			c	d					$2b$	$2b$			$4b$	2
f					e	f					$2b$	$2b$	$4b$	2

$$a = b + c \quad d = 2b \quad 2c = e + f \quad 2d = a + b \quad 2e = c + d$$
$$f = e \quad a = 3b \quad c = d = e - f = 2b$$

$$\left(\frac{k}{2}\right)^3 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{3}\right)^3 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{k}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 < 1$$

$$k^5 < 2^{10} \times 3^3 \implies k < 7.7327$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{7n+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

Converges to 1 with QR = 0.905.

When coefficient sample is $6k + 5$ it alter generating cases.

$$f(n) = \begin{cases} \frac{kn+1}{2} & \text{if } n \equiv 1 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \\ \frac{kn+1}{2} & \text{if } n \equiv 5 \pmod 6 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

For the coefficient, we can use 5 instead of $6p+5$ to get generating sample. Note that A and E vary depending on what they generate:

ed	Produces						Solved in Terms of b						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a		b	c		e			b	$4b$		$3b$		$8b$	4
b				d						$2b$			$2b$	1
c	a		c			f	$4b$		$4b$			$4b$	$12b$	6
d		b			e			b			$3b$		$4b$	2
e			c	d					$4b$	$2b$			$6b$	3
f	a					f	$4b$					$4b$	$8b$	4

$$2a = b + c + e \quad 2b = d \quad 2c = a + f \quad 2d = b + e \quad 2e = c + d$$
$$f = a \quad a = c = f = 4b \quad d = 2b \quad e = 3b$$

$$\left(\frac{k}{2}\right)^4 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{3}\right)^6 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{k}{2}\right)^3 \times \left(\frac{1}{2}\right)^4 < 1$$

$$\implies k^7 < 2^{14} \times 3^6 \implies k < 10.257$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6, \\ \frac{5n+1}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6. \end{cases}$$

Converges to 1 with QR = 0.48747

Example 21:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \text{ —3/6}, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \text{ —1/6}, \\ \frac{kn+3}{2} & \text{if } n \equiv 1 \pmod 6 \text{ or } n \equiv 5 \pmod 6 \text{ —2/6}. \end{cases}$$

$$f(n) = \begin{cases} \frac{kn+3}{2} & \text{if } n \equiv 1 \pmod 6 \text{ —A(c,f)}, \\ \frac{n}{2} & \text{if } n \equiv 2 \pmod 6 \text{ —B(a,d)}, \\ \frac{n}{3} & \text{if } n \equiv 3 \pmod 6 \text{ —C(a,c,e)}, \\ \frac{n}{2} & \text{if } n \equiv 4 \pmod 6 \text{ —D(b,e)}, \\ \frac{kn+3}{2} & \text{if } n \equiv 5 \pmod 6 \text{ —E(c,f)}, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 6 \text{ —F(c,f)}. \end{cases}$$

	Generates						After solved in terms of 1/e						Sum	Simplified
	A	B	C	D	E	F	A	B	C	D	E	F		
a		b	c						2				2	2
b				d									0	0
c	a		c		e	f	1		2		1	2	6	6
d		b											0	0
e			c	d					2				2	2
f	a				e	f	1				1	2	4	4

2a=b+c 2b=d 2c=e+f 2d=b 2 e=c+d f=a+e b=0 d=0 a=e c=2e f=2e

Note:- if a sequence is semi-cycled or a case is not generated it is not considered as Kaakuma sequence

Example 22

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ kn + c & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

We split cases that are partially generated to avoid complexity:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 2 \pmod 4 - A(d,e,f,g) \\ \frac{n}{4} & \text{if } n \equiv 4 \pmod 8 - B(d,e,f,g) \\ \frac{n}{8} & \text{if } n \equiv 0 \pmod 8 - C(\text{all}) \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 - D(b,c) \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 - E(a,b,c) \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 - F(\text{all}) \\ kn + c & \text{if } n \equiv 1 \pmod 8 - G(c) \end{cases}$$

	Generates							After solving in terms of 1/ f						
	A	B	C	D	E	F	G	A	B	C	D	E	F	G
a			$2c$		$2e$	$2f$				30/7		4	2	
b			c	d	e	f				15/7	4	2	1	
c			c	d	e	f	g			15/7	4	2	1	8
d	a	b	c			f		18/7	16/7	15/7			1	
e	a	b	c			f		18/7	16/7	15/7			1	
f	a	b	c			f		8/7	16/7	15/7			1	
g	a	b	c			f		18/7	16/7	15/7			1	
sum								72/7	64/7	120/7	8	8	8	8
simplified								9	8	15	7	7	7	7

$$g = a + b + c + f \quad 7f = a + b + c \quad 4e = a + b + c + f$$

$$2d = a + b + c + f \quad 7c = d + e + f + g \quad 4b = c + d + e + f$$

$$2a = c + e + f$$

when we solve it in terms of f

$$g = 8f \quad e = 2f \quad d = 4f \quad c = 15f/7 \quad b = 16f/7 \quad a = 18f/7$$

$$(1/2)^9 \times (1/4)^8 \times (1/8)^{15} \times (1/2)^7 \times (1/4)^7 \times (1/8)^7 \times k^7 < 1$$
$$\Rightarrow \quad k^7 < 2^{112} \quad \Rightarrow \quad k < 2^{16}$$

When we substitute k:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 7 \pmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 5 \pmod 8 \\ \frac{n+5}{8} & \text{if } n \equiv 3 \pmod 8 \\ 65535n - 65519 & \text{if } n \equiv 1 \pmod 8 \end{cases}$$

The sequence Converges to 1 for all $n \in \mathbb{N}$, $QR = 65535/65536$.

Note: This is a complicated form a sequence in Example 10 where case2 and case5 generate case1 partially.

All these different types of examples show how Qodaa Ratio Test Works even in complicated equations. Qodaa ratio test is simple and rigor to apply. Beyond this there are some points to study in future like number of cycles, interval of constants a sequence to converges, where diverging will start for a diverging kaakuma sequence.

4.3 Proof 3: Computational Analysis

Even though computational analysis cannot serve as a rigorous proof of the Collatz conjecture, it can provide convincing evidence until more rigorous proofs, like Proof 1 and Proof 2, are available. In some challenging cases, and based on their argument level, computational results must be considered, at least to some extent.

4.3.1 Constants and Bounded Values

There are several distinct constants and bounded values observed in the Collatz sequence as discussed in Behavior 3.5.

The average stopping time of the Collatz sequence is a constant, similar to the constants π and e. The function $f(n)$ is defined as:

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

The average stopping time of this sequence is approximately 3.49269. The key point is that if the average stopping time is constant and consistent with very small variation on both sides, it is almost impossible to divert from this behavior after 10^{20} or 10^{40}. If the Collatz conjecture were invalid, this would imply that for 2^{120}, the stopping time t would not align as:

$$(\sum_{n=2}^{2^{120}-1} t)/(2^{120}-1) = 3.49269$$

but:

$$(\sum_{n=2}^{2^{120}} t)/2^{120} = \infty$$

which is impossible.

4.3.2 Inverse Map of Collatz Sequence

The inverse map of the Collatz sequence covers all natural numbers starting from root 1. During this process, its expansion rate is 33.33

4.3.3 Ratio of Stopping Time

The ratio of stopping time to $\log_2(n)$ is bounded and less than 5.5. It is also bounded and less than 5, and small numbers such as 28 and 32 can be adjusted by translation. This can be verified by computer programs using high-rate stopping time values like 2^k. This constant is analogous to the ratio of primes in natural numbers, $\pi(x)$. For example:

$$2^k \quad \frac{4 \times (2^{6k}-1)}{9} \quad \frac{8 \times (2^{18k}-1)}{27} \quad \frac{16 \times (2^{54k}-1)}{81} \quad \frac{32 \times (2^{162k}-1)}{243}$$

4.3.4 Expected Huge Iterations

In the Collatz sequence, it is not surprising to encounter relatively high iteration numbers. As seen in Behavior 3.8, numbers with powers of 2 have relatively high iterations, and numbers that reach powers of 2 before decreasing from the starting numbers also have high iterations. Numbers less than 2^{200000} are expected to have relatively high iterations, and the constants are kept as described in Behavior 3.5.3 and 3.5.4.

4.3.5 Special and Extreme Contradiction in Cycle Case

In the cycle case, the number of iterations needed to create a cycle is $n/10$. If 10^{20} is the first number to create a non-trivial cycle, it must have 10^{19} iterations to the minimum, as discussed in Behavior 3.7. This is contradictory because, based on Analysis 3.5.2, it should only be up to $5.5 \times 60 = 330$ at the maximum.

4.3.6 Collatz Sequence with Falling Values

If there exists a non-Collatz number, its sequence must include iteration group numbers or falling values like $2^{59}k + 28$, $2^{54}k + 64$. These falling values lead to other falling points and make the sequence excessively dense.

4.3.7 Infinite Paradigm-Shifting Kaakuma Sequence

An example of an infinite paradigm-shifting Kaakuma sequence is given by $65535n - 327667$. As seen in Proof 1 Example 10 and Example 25, this sequence has over 2 billion iterations and a height greater than 10^{80}. It takes 15 days to complete iterations for a small number, 9757. This is a highly paradigm-shifting example of a Kaakuma sequence, with many more such cases existing.

Conclusion

The Collatz conjecture is considered true because of the following reasons:

1. Contradiction in tree size balance
2. Qodaa Ratio Test
3. Computational Analysis

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