A short proof of Fermat’s Last Theorem
based on the difference in volume
of two cubes

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Abstract

Over the centuries, numerous mathematicians have tried to proof Fermat’s Last Theorem. In the year 1994, Fermat’s Last Theorem in the form of $a^m + b^m = c^m$ with $a$, $b$ and $c$ being natural numbers and $m$ being a natural number $> 2$ was shown to be correct. In this publication I demonstrate that the difference in volume of two cubes having different side lengths cannot be a cube in itself with a side length having the value of a natural number. This also holds for cubes having higher dimensions than three, since the surfaces of these cubes all consist of three-dimensional cubes.

**Proof for Three-Dimensional Cubes**

In the year 1994, the equation

$$a^m + b^m = c^m \quad (A1a)$$

or

was proven not to have a solution for $m > 2$ and element of naturals, when $a$, $b$ and $c$ are all natural numbers, i.e.

$$a^m + b^m \neq c^m \quad (A1b)$$

(Fermat’s Last Theorem), on over 90 pages. It is known that Fermat himself envisaged a short proof which, however, has never been found in his records.

In the following, I present a short proof of his last theorem based on the difference in volume of two cubes having different side lengths.

We rearrange $(A1a)$ to

$$a^m - c^m = b^m \quad (A2a)$$

and show

$$a^m - c^m \neq b^m \quad (A2b)$$
with $m > 2$ being an element of naturals and also $a, b$ and $c$ being all natural numbers,

With $a$ and $c > a$ being naturals (I) in equation (A2a), the following can be defined:

\[ a^3 = A \quad \text{(volume of a cube with } a \text{ being the side length)} \quad \text{and} \]
\[ c^3 = C \quad \text{(volume of a larger cube with } c \text{ being the side length)} \]

Since $c > a$, $c$ can be expressed as follows (see Fig. 1 below):

\[ c = a + x, \quad \text{(II) with } x < c, \text{ namely } x = c - a \quad \text{and element of the naturals.} \]

Then we get

\[ c^3 = (a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 = C \quad \text{(III) and} \]
\[ 3\sqrt[3]{C} = c = 3\sqrt[3]{(a^3 + 3a^2x + 3ax^2 + x^3)}, \]

and furthermore

\[ c^3 - a^3 = B \quad \text{(IV)}, \]

wherein $B$ is the difference of the volumes of cube $C$ and cube $A$.

We then define

\[ 3\sqrt[3]{B} = b, \text{ with } b \text{ being the side of cube } B, \]

and thus

\[ b^3 = B \]

We then can write:

\[ c^3 - a^3 = b^3 = 3a^2x + 3ax^2 + x^3 \quad \text{(V), which follows from (III) and (IV).} \]

Obviously,

\[ b^3 > x^3 \text{ and } b > x \text{ (see also Fig. 1 below)} \]

Accordingly

\[ b - x > 0 \]

We now define
\[ b - x = y \] and thus
\[ b = x + y \ (\text{VI}), \text{ wherein } y \text{ is at least a positive real number.} \]

Accordingly,
\[ b^3 = B = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \ (\text{VII}) \]

On the other hand,
\[ b^3 = 3a^2x + 3ax^2 + x^3 \ (\text{V}) \]

We now can set up the equation \((\text{VII}) = (\text{V})\)
\[ x^3 + 3x^2y + 3xy^2 + y^3 = 3a^2x + 3ax^2 + x^3 \ (\text{VIII}) \]

and examine, if \( b = x + y \) can be a natural.

If \( b \) is to be a natural, also \( y \) has to be a natural, since \( x \) according to \((\text{II})\) is a natural.

Conversion of \((\text{VIII})\) delivers:
\[ 3x^2y + 3xy^2 + y^3 = 3a^2x + 3ax^2 \]
\[ y^3 + 3x^2y + 3xy^2 - 3x^2a - 3xa^2 = 0 \ (\text{IX}); \]

This is a polynomial of third degree, which is notoriously difficult to solve.

However, \((\text{IX})\) is also a quadratic equation of \( x \), which is considerably easier to solve than a polynomial of third degree.

\((\text{IX})\) solved for \( x \) gives:
\[ (3a-3y)x^2 + (3a^2-3y^2)x - y^3 = 0 \ (\text{X}) \]
\[ (a-y)x^2 + (a^2-y^2)x -y^3/3 = 0 \]
\[ x_1, x_2 = \frac{-(a^2-y^2) +/- \sqrt{(a^2-y^2)^2 - 4(a-y)(-y^3/3))}}{2(a-y)} \ (\text{XI}) \]
\[ = \frac{-(a^2-y^2) +/- \sqrt{(a^4-2a^2y^2+y^4 - 4(a-y)(-y^3/3))}}{2(a-y)} \]
\[ = \frac{-(a^2-y^2) +/- \sqrt{(a^4-2a^2y^2+y^4 - 4y^4/3+4ay^3/3))}}{2(a-y)} \ (\text{XII}) \]

or further converted
\[ = \frac{-(a^2-y^2) +/- \sqrt{(a^4+y^4-y^2/3 + 4ay^3/3-2a^2y^2))}}{2(a-y)} \ (\text{XII}) \]
\[ = \frac{-(a^2-y^2) +/- \sqrt{(a^4-y^2/3(y^2 - 4ay + 6a^2))}}{2(a-y)} \ (\text{XIII}) \]

\( x \) in \((\text{XI})\) und \((\text{XII})\) is expressed as a function of \( y \), which according to \((\text{II})\) has to deliver \( x \) as a natural, if equation \((\text{VIII})\) were to yield \( x+y = b \) with \( b \) being the side of a cube as
a natural. This means that the function for x may in no case contain an irrational or complex number, and more specifically that the expression under the square root as a whole may not yield an irrational or complex number, nor y as an irrational or complex number.

The total expression under the square root can be natural or rational only in three instances:

1) If the total expression could be converted to \(((s^2a^2-t^2y^2)^2\) with s and t being optional fractions; this is obviously not the case.

2) If the total expression under the square root could be converted to \((sa+ty)^4\), with s and t having the same meanings as above. This, too, is obviously not the case, since \(y^4/3\) is not the fourth potency of a natural or rational number.

3) If the expression \(-y^2/3(y^2 + 4ay-6a^2)\) were set to zero, since then only \(\sqrt{a^4}\) remains. This can be done in two ways. One is to set y to 0, but this contradicts prerequisite (VI). The other one is to set

\[(y^2 - 4ay + 6a^2) = 0\]

With this we get

\[y_{1,2} = [4a +/- \sqrt{(16a^2 - 24a^2)}/2 = 2a +/- a\sqrt{-2}\]

Thus, the square root yields an natural, but at the cost of y being a complex number. If this solution for y is put into

\[x_{1,2} = \frac{-(a^2-y^2) +/- \sqrt{(a^4-y^2/3(y^2 - 4ay + 6a^2))}}{2(a-y)} \] (XIII)

we get:

\[x_{1,2} = \frac{-(a^2-y^2) +/- \sqrt{a^4}}{2(a-y)},\]

and with

\[y_{1,2} = 2a +/- a\sqrt{-2}\]

we get

\[x_{1,2} = \frac{-(a+2a +/- a\sqrt{-2})( a-(2a +/- a\sqrt{-2})) +/- a^2}{2(a-2a +/- a\sqrt{-2})}\]

\[= \frac{-(a+2a +/- a\sqrt{-2})}{2 +/- a^2/2(a-2a +/- a\sqrt{-2})}\]

\[= \frac{-(a+2 +/- a\sqrt{-2})}{2 +/- a^2/2a(1-2 +/- a\sqrt{-2})}\]

\[= \frac{-(a/2)(3 +/- a\sqrt{-2})}{2(1-2 +/- a\sqrt{-2})}\]

\[= \frac{-(a/2)(3 +/- a\sqrt{-2})}{2(-1 +/- a\sqrt{-2})}\]
\[
= -\frac{3a}{2} +/- a\sqrt{-2/2} +/- a[-2 +/- 2\sqrt{-2}]
\]
\[
= -\frac{3a}{2} +/- (a\sqrt{2}i/2) +/- a[-2 +/- (2\sqrt{2})i]
\]

Thus, \(x_{1,2}\) are also complex numbers, when \(y_{1,2}\) are complex numbers.

According to the above, we showed that there is no solution for \(b\) in (VI), wherein \(b = x + y\) (VI) is a natural, and accordingly there is no solution for (A1a) or (A2a), in which \(m = 3\) and all three of \(a\), \(b\) and \(c\) are naturals,

Since the surfaces of all cubes of dimensions higher than three consist of three-dimensional cubes, the above also proves Fermat's Last Theorem for all \(m>3\).