A BOUND FOR THE ISOTROPIC CONSTANT

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Abstract. We obtain a dimension-independent bound for the isotropic constant for convex bodies. A key idea is trying to find the right scale for the problem when considering John's position.

Contents

1. Introduction

The isotropic conjecture or the Bourgain's slicing problem asks the existence of a following universal constant c.

Theorem 1.1. *There exists an affine hyperplane H and an universal constant c such that*

$$m_{n-1}(H \cap K) > c,$$

for convex bodies K of unit volume.

A classic reference for these kind of questions is [9]. More recently the claim is already proved up to a polylog with very modern methods [6].Those methods very introduced in the groundbreaking work by Chen [5]. The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i}{|K|}\frac{\int_K x_j}{|K|}.$$

We define the isotropic constant of any convex body K in scaling invariant way using

$$L_K^{2n} := \frac{\mathrm{Det}(CovK)}{|K|^2}.$$

The isotropic position is a position, when the covariance matrix is diagonal and all the diagonal entries are the same. Moreover, it is assumed that the volume is unit. This kind of position exists [9]. An another position that always exists is the John's position. It is the position of a convex body, where the minimal circumscribed ellipsoid is the unit ball. We prove the Bourgain's slicing conjecture by proving an universal upper bound for the isotropic constant.

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2. Previously known results

For any measurable set A we let $|A|$ be the n-dimensional Lebesque measure.The inner volume ratio for a convex body K is defined as

$$ivr(K) := \min_T (|K|/|T(B_n)|)^{1/n},$$

where T is an affine map, B_n the standard unit ball and $T(B_n) \subset K$. The outer volume ratio for a convex body K is defined as

$$ovr(K) := \min_T (|T(B_n)|/|K|)^{1/n}, \tag{2.1}$$

where T is an affine map, B_n the standard unit ball and $K \subset T(B_n)$. Ball [2] and Barthe [4] proved using the Braschamb-lieb [8] and reversed Braschamb-Lieb [4] inequalities, respectively, that in the non-symmetric case $ivr(K)$ and $ovr(K)$ are maximized when the convex body K is the standard simplex S_n. Moreover, in the symmetric case $ivr(K)$ is maximized when K is the cube C_n and $ovr(K)$ is maximized when K is the crosspolytope CP_n. The extended Khinchine inequality says that for any convex bodies

$$(\frac{1}{|K|}\int_K |x_i|^2 dx)^{1/2} \leq C\frac{1}{|K|}\int_K |x_i| dx. \tag{2.2}$$

A proof can be found in [7].

3. The proof

First we show a key fact.

Theorem 3.1. *Let K be a convex body in a John's position. Then*

$$|K|^{1/n} \geq c'(n!)^{-1/n} > cn^{-1}.$$

Proof. For K in John's position we have that $K \subset B(0,1)$.

Remark 3.2. for the diameter d we have that

$$1 \leq d \leq 2.$$

Via (2.1) we have that

$$\frac{|B(0,1)|}{|S_n|} \geq \frac{|B(0,1)|}{|K|}.$$

So

$$\frac{1}{|S_n|} \geq \frac{1}{|K|}.$$

Thus,

$$|S_n| \leq |K|.$$

Now, we just need to calculate the volume of the standard simplex S_n in John's position. We have that

$$|S|^{1/n} > Cn^{-1}, \tag{3.1}$$

where C is an universal constant. So (3.1) gives us the claim. □

We will also need the lemma:

Lemma 3.3. *Let K be a convex body. If $||x||_2 \le a$ then*

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \le \frac{(n+1)}{n} \int_{B(0,a)} \frac{|x_i| dx}{|B(0,a)|}.$$

Proof. We have from Cauchy-Schwarz and isotropy that

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \le \int_K \frac{||x||_2 dx}{\sqrt{n}|K|} \le \frac{a}{\sqrt{n}}.$$

On the other hand we have

$$\int_{B(0,a)} \frac{|x_i|^2 dx}{|B(0,a)|} = \int_{B(0,a)} \frac{||x||_2 dx}{n|B(0,a)|} = \frac{a\sqrt{n}}{(n+1)}.$$

□

The following theorem is the key theorem.

Theorem 3.4. *It holds in a scaled John's position that*

$$\frac{1}{n}\sum_{i=1}^n \left(\int_K \frac{|x_i|^2}{|K|^{1+2/n}} dx\right)^{1/2} \le C. \tag{3.2}$$

Proof. It's clear that the inequality (3.2) is scaling invariant. Thus, w.l.o.g let K be a convex body in a scaled John's position such that

$$|K|^{1/n} = \frac{1}{\sqrt{n}}. \tag{3.3}$$

We have from theorem 3.1 via scaling, and from above (3.3) that

$$\frac{1}{\sqrt{n}} = |K|^{1/n} \ge ac'n^{-1},$$

where a is the smallest possible. Thus,

$$a \le c\sqrt{n}. \tag{3.4}$$

Then, we have from the equality (3.3), inequality (3.4), and from the lemma 3.3 that

$$\int_K \frac{||x||_1 dx}{n|K|^{1+1/n}} \le \int_K \frac{\sqrt{n}|x_i| dx}{n|K|} \le \int_K \frac{|x_i| dx}{\sqrt{n}|K|} \le \frac{(n+1)}{n} \int_{B(0,c\sqrt{n})} \frac{|x_i| dx}{|B(0,c\sqrt{n})|}$$
$$= \frac{(n+1)}{n^{3/2}} \int_{B(0,c\sqrt{n})} \frac{||x||_2 dx}{|B(0,c\sqrt{n})|} \le \frac{c\sqrt{n}}{\sqrt{n}} = c.$$

Thus,

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|^{1+1/n}} \le c. \tag{3.5}$$

We have then using the extended Khinchine, and from inequality 3.5 that

$$\frac{1}{n}\sum_{i=1}^{n}(\int_K \frac{|x_i|^2dx}{|K|^{1+2/n}})^{1/2} \le (\int_K 1_{|x|\le 1}\frac{|x_i|^2dx}{|K|^{1+2/n}})^{1/2} + \frac{1}{n}\sum_{i=1}^{n} C\int_K 1_{|x|>1}\frac{|x_i|dx}{|K|^{1+1/n}}$$
$$\le (\int_K 1_{|x|\le 1}\frac{|x_i|^2dx}{|K|^{1+2/n}})^{1/2} + C\int_K \frac{||x||_1dx}{n|K|^{1+1/n}}$$
$$\le (\int_K 1_{|x|\le 1}\frac{|x_i|^2dx}{|K|^{1+2/n}})^{1/2} + C$$
$$\le C\sqrt{n}(\int_K \frac{||x||_2^2 1_{|x|\le 1}dx}{n|K|})^{1/2} + C$$
$$\le C\sqrt{n}(\int_K \frac{dx}{n|K|})^{1/2} + C \le C + C \le C'.$$

□

We can assume that the covariance matrix is diagonal, because it is real and symmetric. So it can be diagonalized by an orthogonal matrix. Because K is centralized, we have

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|}.$$

Moreover, we assume K is in a John's position. We have

$$L_K^n = (\prod_{i=1}^{n}\int_K \frac{x_i x_i dx}{|K|^{1+2/n}})^{1/2}.$$

Now, after taking the nth root we have

$$\begin{aligned} L_K &= (\prod_{i=1}^{n}\int_K \frac{|x_i|dx}{|K|^{1+1/n}})^{1/(2n)} \\ &\le \frac{1}{n}\sum_{i=1}^{n}(\int_K \frac{|x_i|^2dx}{|K|^{1+2/n}})^{1/2} \\ &\le C, \end{aligned}$$

where we used the GM-AM inequality, and the theorem 3.4.This ends the proof of the theorem 1.1.

References

1. Ball, K. *An Elementary Introduction to Modern Convex Geometry* In: Flavors of Geometry, Math. Sci.Res. Inst. Publ., 31, Cambridge University Press, Cambridge (1997).
2. K.M. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) 44 (1991), 351-359
3. F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. 134 (1998), 335-361.
4. F. Barthe, *An extremal property of the mean width of the simplex*, Math. Ann. 310 (1998), 685-693.
5. Y. Chen, *An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture* Geom. Funct. Anal. (GAFA), Vol. 31, (2021), 34?-61
6. B.Klartag *Logarithmic bounds for isoperimetry and slices of convex sets* Ars Inveniendi Analytica (2023), Paper No. 4, 17 pp.

7. M. Gromov and V.D. Milman, *Brunn theorem and a concentration of volume of convex bodies*, GAFA Seminar Notes, Tel Aviv University, Israel 1983-1984, Exp.V., 12pp.
8. H.J. Brascamp and E.H. Lieb, *Best constants in Young?s inequality, its converse and its generalization to more than three functions*, Adv. in Math. 20 (1976), 151-173

9. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Lecture Notes in Mathematics 1376, Springer, Berlin (1989), 64-104.

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