A BOUND FOR THE ISOTROPIC CONSTANT

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Abstract. We obtain a dimension-independent bound for the isotropic constant for convex bodies. A key idea in the proof is to keep the diameter approximately constant and try the control the mass. Often in these kind of questions the volume is kept constant.

Contents

1. Introduction

The isotropic conjecture or the Bourgain's slicing problem asks the existence of a following universal constant c.

Theorem 1.1. *There exists an affine hyperplane H and an universal constant c such that*

$$m_{n-1}(H \cap K) > c,$$

for convex bodies K of unit volume.

A classic reference for these kind of questions is [9]. More recently the claim is already proved up to a polylog with very modern methods [6].Those methods very introduced in the groundbreaking work by Chen [5]. The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i}{|K|}\frac{\int_K x_j}{|K|}.$$

We define the isotropic constant of any convex body K in scaling invariant way using

$$L_K^{2n} := \frac{\mathrm{Det}(CovK)}{|K|^2}.$$

The isotropic position is a position, when the covariance matrix is diagonal and all the diagonal entries are the same. Moreover, it is assumed that the volume is unit. This kind of position exists [9]. An another position that always exists is the John's position. It is the position of a convex body, where the minimal

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circumscribed ellipsoid is the unit ball. We prove the Bourgain's slicing conjecture by proving an universal upper bound for the isotropic constant.

2. Previously known results

For any measurable set A we let $|A|$ be the n-dimensional Lebesque measure.The inner volume ratio for a convex body K is defined as

$$ivr(K) := \min_T (|K|/|T(B_n)|)^{1/n},$$

where T is an affine map, B_n the standard unit ball and $T(B_n) \subset K$. The outer volume ratio for a convex body K is defined as

$$ovr(K) := \min_T (|T(B_n)|/|K|)^{1/n}, \tag{2.1}$$

where T is an affine map, B_n the standard unit ball and $K \subset T(B_n)$. Ball [2] and Barthe [4] proved using the Braschamb-lieb [8] and reversed Braschamb-Lieb [4] inequalities, respectively, that in the non-symmetric case $ivr(K)$ and $ovr(K)$ are maximized when the convex body K is the standard simplex S_n. Moreover, in the symmetric case $ivr(K)$ is maximized when K is the cube C_n and $ovr(K)$ is maximized when K is the crosspolytope CP_n. The extended Khinchine inequality says that for any convex bodies

$$(\frac{1}{|K|}\int_K |x_i|^2 dx)^{1/2} \leq C\frac{1}{|K|}\int_K |x_i| dx. \tag{2.2}$$

A proof can be found in [7].

3. The proof

First we show a key fact.

Theorem 3.1. *Let K be a convex body of unit diameter in a scaled John's position. Then*

$$|K|^{1/n} \geq c'(n!)^{-1/n} > cn^{-1}.$$

Proof. For K' in John's position we have that $K' \subset B(0,1)$. So for the diameter d we have that

$$1 \leq d \leq 2.$$

Moreover, via (2.1) we have that

$$\frac{|B(0,1)|}{|S_n|} \geq \frac{|B(0,1)|}{|K'|}.$$

So

$$\frac{1}{|S_n|} \geq \frac{1}{|K'|}.$$

Thus,

$$|S_n| \leq |K'|.$$

Now, the diameter of K was the unit. So we have

$$|S_n| \leq 2^n|K|.$$

Thus,

$$|S_n|^{1/n} \leq 2|K|^{1/n}. \tag{3.1}$$

Now, we just need to calculate the volume of the standard simplex S_n in John's position. We have that

$$|S|^{1/n} > Cn^{-1}, \tag{3.2}$$

where C is an universal constant. So combining (3.1) and (3.2) gives us the claim.

□

We will also need the lemma showing the essential monotonicity of the means.

Lemma 3.2. *Let K be a convex body. If $||x||_2 \leq a$ then*

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \leq C \int_{B(0,a)} \frac{|x_i| dx}{|B(0,a)|}.$$

Proof. We have

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \leq \int_K \frac{\sqrt{n}||x||_2 dx}{n|K|} \leq \frac{a}{\sqrt{n}}.$$

On the other hand we have

$$\int_{B(0,a)} \frac{|x_i|_2 dx}{|B(0,a)|} = \int_{B(0,a)} \frac{||x||_2 dx}{\sqrt{n}|B(0,a)|} = \frac{an}{(n+1)\sqrt{n}}.$$

□

The following theorem is the key theorem.

Theorem 3.3. *Let K be a convex body in a scaled John's position such that*

$$\int_K ||x||_1 dx = |K|. \tag{3.3}$$

Then it holds in a scaled John's position that

$$\int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i| dx}{|K|^{1+1/n}} \leq C. \tag{3.4}$$

Proof. We notice that the diameter of K must be greater than a constant. Assuming that $||x||_2 \leq a$ we have from the essential monotonicity of the means (3.2), that

$$\frac{1}{n} = (\frac{\int_K \sum_{i=1} |x_i| dx}{n|K|} \leq \frac{C\int_{B(0,a)} |x_i| dx}{n|B(0,a)|} = \frac{Ca}{(n+1)}.$$

Then little algebra gives us

$$a > c.$$

Remark 3.4. It's clear that the position (3.3) exists because the average can be the unit.

So we have from theorem 3.1 that

$$|K|^{1/n} \geq c' n^{-1}. \tag{3.5}$$

Thus, we have

$$\begin{aligned} &\int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i|}{|K|^{1+1/n}} dx \\ &\leq cn \int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i|}{|K|} dx \\ &= c, \end{aligned}$$

where we used the inequality (3.5) and the asumption (3.3). □

We can assume that the covariance matrix is diagonal, because it is real and symmetric. So it can be diagonalized by an orthogonal matrix. Because K is centralized, we have

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|}.$$

Moreover, we assume K is in a John's position. We have

$$\begin{aligned} L_K^n &= (\prod_{i=1}^n \int_K \frac{x_i x_i dx}{|K|^{1+2/n}})^{1/2} \\ &= \prod_{i=1}^n (\int_K \frac{|x_i|^2 dx}{|K|^{1+2/n}})^{1/2} \\ &\leq \prod_{i=1}^n C \int_K \frac{|x_i| dx}{|K|^{1+1/n}}, \end{aligned}$$

where we used the extended Khinchine's inequality (2.2). Now, after taking the nth root we have

$$\begin{aligned} L_K &= (\prod_{i=1}^n C \int_K \frac{|x_i| dx}{|K|^{1+1/n}})^{1/n} \\ &\leq \frac{C}{n} \sum_{i=1}^n \int_K \frac{|x_i| dx}{|K|^{1+1/n}} \\ &\leq C, \end{aligned}$$

where we used the GM-AM inequality and the theorem 3.3. It's clear that the inequality (3.4) is scaling invariant. This ends the proof of the theorem 1.1.

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