No blowup for the Navier–Stokes equations

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in \( \mathbb{R}^3 \), see [1,3]. Let \( u = u(x,t) \in \mathbb{R}^3 \) be the fluid velocity and let \( p = p(x,t) \in \mathbb{R} \) be the fluid pressure, each dependent on position \( x \in \mathbb{R}^3 \) and time \( t \geq 0 \). I take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity \( \nu > 0 \) and to fill all of \( \mathbb{R}^3 \). The Navier–Stokes equations can then be written as

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nu \nabla^2 u - \nabla p, \tag{1}
\]

\[
\nabla \cdot u = 0 \tag{2}
\]

with initial condition

\[
u(x, 0) = u^0 \tag{3}
\]

where \( u^0 = u^0(x) \in \mathbb{R}^3 \). In these equations

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{4}
\]

is the gradient operator and

\[
\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \tag{5}
\]

is the Laplacian operator. When \( \nu = 0 \) equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

\[
u^0(x + e_i) = u^0(x) \tag{6}
\]

for \( 1 \leq i \leq 3 \) where \( e_i \) is the \( i \)th unit vector in \( \mathbb{R}^3 \). The initial condition \( u^0 \) is a given \( C^\infty \) divergence-free vector field on \( \mathbb{R}^3 \). A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

\[
u(x + e_i, t) = u(x, t), \quad p(x + e_i, t) = p(x, t) \tag{7}
\]

on \( \mathbb{R}^3 \times [0, \infty) \) for \( 1 \leq i \leq 3 \) and

\[
u, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{8}
\]
2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem, see [2,3,5,6].

**Theorem.** Take $\nu > 0$. Let $u^0$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $u, p$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).

**Proof.** Let $u, p$ be given by

$$u = \sum_{L=\infty}^{\infty} u_L e^{i k_L \cdot x}, \quad (9)$$

$$p = \sum_{L=\infty}^{\infty} p_L e^{i k_L \cdot x}, \quad (10)$$

respectively. Here $u_L = u_L(t) \in \mathbb{C}^3$, $p_L = p_L(t) \in \mathbb{C}$, $i = \sqrt{-1}$, $k = 2\pi$, and $\sum_{L=\infty}^{\infty}$ denotes the sum over all $L \in \mathbb{Z}^3$. The initial condition $u^0$ is a Fourier series [2] of which is convergent for all $x \in \mathbb{R}^3$. Equation (1) implies

$$\sum_{L=\infty}^{\infty} \frac{\partial u_L}{\partial t} e^{i k_L \cdot x} + \sum_{L=\infty}^{\infty} \sum_{M=\infty}^{\infty} (u_L \cdot i k M) u_M e^{i k (L+M) \cdot x} = - \sum_{L=\infty}^{\infty} \nu k^2 |L|^2 u_L e^{i k L \cdot x} - \sum_{L=\infty}^{\infty} i k L p_L e^{i k L \cdot x}. \quad (11)$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial u_L}{\partial t} + \sum_{M=\infty}^{\infty} (u_{L-M} \cdot i k M) u_M = -\nu k^2 |L|^2 u_L - i k L p_L \quad (12)$$

on using the Cauchy product type formula [4]

$$\sum_{l=\infty}^{\infty} a_l x^l \sum_{m=\infty}^{\infty} b_m x^m = \sum_{l=\infty}^{\infty} \sum_{m=\infty}^{\infty} a_{l-m} b_m x^l. \quad (13)$$

Equation (2) implies

$$\sum_{L=\infty}^{\infty} i k L \cdot u_L e^{i k L \cdot x} = 0. \quad (14)$$

Equating like powers of the exponentials in (14) yields

$$L \cdot u_L = 0. \quad (15)$$
Applying $\mathbf{L} \cdot$ to (12) and noting (15) leads to

$$p_L = - \sum_{M=-\infty}^{\infty} (\mathbf{u}_{L-M} \cdot \hat{\mathbf{L}})(\mathbf{u}_M \cdot \hat{\mathbf{L}})$$  

(16)

where $p_0$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_L}{\partial t} = - \sum_{M=-\infty}^{\infty} (\mathbf{u}_{L-M} \cdot iM)\mathbf{u}_M - \nu k^2|\mathbf{L}|^2\mathbf{u}_L + \sum_{M=-\infty}^{\infty} iM(\mathbf{u}_{L-M} \cdot \hat{\mathbf{L}})(\mathbf{u}_M \cdot \hat{\mathbf{L}})$$  

(17)

where $\mathbf{u}_0 = \mathbf{u}_0(0)$. Without loss of generality [2], I take $\mathbf{u}_0 = 0$. This is due to the Galilean invariance property of solutions to the Navier–Stokes equations. The equations for $\mathbf{u}_L$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^3$. Here we can find a representation of the solution $\mathbf{u}, p$ and show that $\mathbf{u}$ can not have a finite time singularity when $\mathbf{u}^e(x)$ is smooth.

First note that the solution $\mathbf{u} = \mathbf{u}(x, t)$ to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0,$$  

(18)

$$\mathbf{u}(x, 0) = \mathbf{u}^e(x)$$  

(19)

can be represented by

$$\mathbf{u} = \mathbf{u}^e(\mathbf{X}),$$  

(20)

$$\mathbf{X} = \mathbf{x} + t\mathbf{u}^e(\mathbf{X}).$$  

(21)

This can be checked as follows via the chain rule.

$$\frac{\partial \mathbf{u}^e(\mathbf{X})}{\partial t} = \frac{\partial \mathbf{u}^e(\mathbf{X})}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial t} = \frac{\partial \mathbf{u}^e(\mathbf{X})}{\partial \mathbf{X}} - \mathbf{u}^e(\mathbf{X})$$

$$= - \frac{\partial \mathbf{u}^e(\mathbf{X})}{\partial \mathbf{x}}\mathbf{u}^e(\mathbf{X}).$$  

(22)

Therefore (18), (19) are satisfied. We see here that this $\mathbf{u}$ satisfying (18), (19) can not have a finite time singularity when $\mathbf{u}^e(x)$ is smooth.

The solution $\mathbf{u} = \mathbf{u}(x, t), p = p(x, t)$ to (1), (2), (3), (6), (7) can be represented by

$$\mathbf{u} = \mathbf{u}^e(\mathbf{X}),$$  

(23)

$$p = -\nabla_x^2[\nabla_x \cdot [(\mathbf{u}^e(\mathbf{X}) \cdot \nabla_x)\mathbf{u}^e(\mathbf{X})]] = P(\mathbf{X}),$$  

(24)

$$\mathbf{X} = \mathbf{x} + t[\mathbf{u}^e(\mathbf{X}) - \mathbf{u}^e(\mathbf{X})]$$  

(25)
where \( u'(X) \) is a representation of the implicit solution \( u^o(X) \) to

\[
- \frac{\partial u^o(X)}{\partial X} u^o(X) = \nu \nabla^2_X u^o(X) + \nabla_X P(X).
\]

(26)

In these equations \( \nabla_X, \nabla^2_X, \) and \( \nabla^{-2}_X \) denote the gradient, Laplacian, and inverse Laplacian with respect to the variable \( X \) respectively. Note it is true that

\[
u u^o(X) = \frac{1}{\nu} \nabla^{-2}_X \left[ -\frac{\partial u^o(X)}{\partial X} u^o(X) - \nabla X P(X) \right].
\]

(27)

It is also true that \( u'(X) \) can be represented by

\[
u u^o(X) = -\left( \frac{\partial u^o(X)}{\partial X} \right)^{-1} [\nu \nabla^2_X u^o(X) + \nabla_X P(X)]
\]

(28)

in cases where the nonlinearity is not identically equal to zero. The solution can be checked as follows via the chain rule.

\[
\frac{\partial u^o(X)}{\partial t} = \frac{\partial u^o(X)}{\partial X} \frac{\partial X}{\partial t} = \frac{\partial u^o(X)}{\partial X} [u^o(X) - u'(X)]
\]

or

\[
\frac{\partial u^o(X)}{\partial X} = \left( \frac{\partial u^o(X)}{\partial X} \right)^{-1} [\nu \nabla^2_X u^o(X) + \nabla X P(X)]
\]

(29)

Therefore (1), (3) are satisfied. We also have

\[
\nu = -\nabla^2 \left[ \nabla \cdot \left( \nu \nabla u^o(X) \right) \right]
\]

(30)

which is obtained by applying \( \nabla \cdot \) to (1) of which ensures (2). Due to the form of the solution we see that \( u \) can not have a finite time singularity when \( u^o(x) \) is smooth. Therefore the theorem is true.

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References
