Geometric Facts of a Circle Through its Equation

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Abstract
We derive equation of a circle passing through three distinct points in a different way and demonstrate that several basic geometric properties of a circle can be derived easily by means of this equation alone.

1 Introduction
In the complex plane, the equation $|z - c| = r$ represents a circle with centre $c$ and radius $r$. This is also known as centre-radius form of a circle. The equation of circle passing through three distinct and non-collinear points $z_1, z_2, z_3$ is generally written as $\arg\left(\frac{z - z_1}{z - z_2} \frac{z - z_3}{z - z_2}\right) = 0$ or $\pi$, see [1]. We here consider centre-radius form of a circle and obtain $c$ and $r$ in terms of the expressions involving $z_1, z_2, z_3$. Then, we demonstrate that some geometric properties of circles and triangles can be derived by means of this equation alone. The underlying thought is that a unique circle and a unique triangle pass through three distinct points and therefore its equation must yield some geometrical properties of circles and the triangles.

2 Equation of Circle
We begin with writing the equation of circle in a form which immediately implies several geometric properties of a circle

Theorem. Let $|z - c| = r$ be the equation of circle passing through $z_1, z_2, z_3$. Let $\alpha = (z_2 - z_3)(\overline{z_1} - \overline{z_3})$. Then,

$$c = \left(1 - i \frac{\text{Re} \alpha}{\text{Im} \alpha}\right) \frac{z_1}{2} + \left(1 + i \frac{\text{Re} \alpha}{\text{Im} \alpha}\right) \frac{z_2}{2}, \quad (1)$$

$$r = \sqrt{1 + \left(\frac{\text{Re} \alpha}{\text{Im} \alpha}\right)^2 \frac{|z_2 - z_1|}{2}}. \quad (2)$$

Proof. We have \( |z_j - c| = r \), therefore, \( f(z_j) = |z_j|^2 + |c|^2 - 2 \text{Re} \, c \bar{z}_j = r^2 \) for, \( j = 1, 2, 3 \). Then, \( f(z_3) - f(z_1) = 0 \) and \( f(z_2) - f(z_3) = 0 \) give two linear equations in \( c \) and \( r \) the solution of which gives (1). Further, for \( |z_1 - c| = |z_2 - c| = r \), we have \( 2r^2 = \sum_{j=1}^{2} |z_i - c|^2 \). Also, \( \sum_{j=1}^{2} (z_i - \frac{z_1 + z_2}{2}) = 0 \). Therefore

\[
r^2 = \frac{1}{2} \sum_{i=1}^{2} \left| z_i - \frac{z_1 + z_2}{2} + \frac{z_1 + z_2}{2} - c \right|^2 = \left| z_2 - z_1 \right|^2 + \left| z_1 + z_2 - c \right|^2. \tag{3}
\]

Inserting (1) in (3) and simplify, we immediately get (2). \( \blacksquare \)

3 Geometric Facts of a Circle

We now demonstrate that the above theorem yields the geometric properties of circles.

(i) The diameter is the largest chord in a circle. From (2), \( |z_2 - z_1| \leq 2r = \text{diameter.} \)

(ii) The line segment joining the mid point of a chord and centre of a circle is perpendicular to the chord. From (1), \( c - \frac{z_1 + z_2}{2} = ik (z_1 - z_2) \) where \( k = -\frac{1}{2} \text{Re} \alpha \) is a real number. It follows that \( c - \frac{z_1 + z_2}{2} \) is perpendicular to \( z_1 - z_2 \).

(iii) Equal chords of a circle are equidistant from the centre of the circle. For any four distinct points \( z_1, z_2, z_3, z_4 \) on \( |z - c| = r \), we have from (3),

\[
\left| c - \frac{z_1 + z_2}{2} \right|^2 = r^2 - \left| \frac{z_1 - z_2}{2} \right|^2 \quad \text{and} \quad \left| c - \frac{z_3 + z_4}{2} \right|^2 = r^2 - \left| \frac{z_3 - z_4}{2} \right|^2. \tag{4}
\]

It follows from (4) that \( \left| c - \frac{z_1 + z_2}{2} \right| = \left| c - \frac{z_3 + z_4}{2} \right| \) if and only if \( |z_1 - z_2| = |z_3 - z_4| \).

(iv) Equal chords of a circle subtend equal angles at the centre of the circle. From (4), \( |z_1 - z_2| = |z_3 - z_4| \) if and only if \( \left| c - \frac{z_1 + z_2}{2} \right| = \left| c - \frac{z_3 + z_4}{2} \right| \) if and only if \( \text{Re} (c - z_1) (\overline{c} - z_2) = \text{Re} (c - z_3) (\overline{c} - z_4) \) if and only if chords subtend equal angles at the centre.

(v) A triangle inscribed in a circle is right triangle if and only if one side of the triangle is the diameter of the circumcircle (Thales’s theorem). From (2), \( |z_2 - z_1| = 2r \) if and only if \( \text{Re} \alpha = 0 \) if and only if triangle is right triangle at \( z_3 \).

(vi) The circumcenter lies on triangle if and only if triangle is right triangle and circumcenter lies on hypotenuse (Thales’s theorem). From (1), we see that \( c = \frac{z_1 + z_2}{2} \) if and only if \( \text{Re} \alpha = 0 \). This implies \( c = \frac{z_1 + z_2}{2} \) if and only if triangle with the vertices \( z_1, z_2, z_3 \) is right triangle at \( z_3 \). Further, \( c = \frac{z_1 + z_2}{2} \) if and only if \( z_1 \) and \( z_2 \) are the end points of the diameter.
(vii) A chord subtends equal angle at all points on a circle on one side of the chord and supplementary angle on other side of the chord (cf. [2, p. 13-14]). Let \( \theta \) be the angle between \( z_2 - z_3 \) and \( z_1 - z_3 \). Then, \( \Re \alpha = |z_2 - z_3||z_1 - z_3| \cos \theta \) and \( \Im \alpha = |z_2 - z_3||z_1 - z_3| \sin \theta \). We therefore have \( \frac{\Re \alpha}{\Im \alpha} = \cot \theta \). Then, the circumcenter (1) can equivalently be written as

\[
c = (1 - i \cot \theta) \frac{z_1}{2} + (1 + i \cot \theta) \frac{z_2}{2}.
\]

(5)

It is clear from (2) that the value of \( \frac{\Re \alpha}{\Im \alpha} \) remain same for any two fixed points \( z_1, z_2 \) on the circle. Hence, \( |\cot \theta| \) is constant for any point \( z_3 \) on the circle.

Let \( z_j = c + re^{i \theta_j}, \ j = 1, 2, 3 \). Then \( \alpha = 4r^2 \sin \frac{\theta_3 - \theta_2}{2} \sin \frac{\theta_2 - \theta_1}{2} e^{i \frac{\theta_2 - \theta_1}{2}} \).

For \( \theta_1 < \theta_2 < \theta_3 < 2\pi \), \( \Im \alpha > 0 \) but \( \Re \alpha > 0 \) when \( \theta_2 - \theta_1 \leq \pi \), otherwise \( \Re \alpha < 0 \). For \( \theta_1 < \theta_3 < \theta_2 < 2\pi \), \( \Im \alpha < 0 \) but \( \Re \alpha < 0 \) when \( \theta_2 - \theta_1 \leq \pi \) otherwise \( \Re \alpha > 0 \). Thus, if \( \theta_2 - \theta_1 \leq \pi \) and \( z_j \)'s are arranged in counterclockwise direction as \( z_1, z_2, z_3 \), we have \( 0 < \theta \leq \frac{\pi}{2} \) (\( \theta = \phi_1 \) say) and if \( z_j \)'s are arranged as \( z_1, z_3, z_2 \), we have \( \frac{\pi}{2} < \theta \leq \pi \) (\( \theta = \phi_2 \)). But \( |\cot \theta| \) is constant, and it follows that \( \phi_1 + \phi_2 = \pi \). The arguments for the case \( \theta_2 - \theta_1 \geq \pi \) are same.

(viii) Sine formula of triangle. We have \( \frac{\Re \alpha}{\Im \alpha} = \cot \theta \), therefore from (2) we find that

\[
r = \sqrt{1 + \cot^2 \theta} \frac{|z_2 - z_1|}{2} = \frac{|z_2 - z_1|}{2 \sin \theta}.
\]

(6)

Further, if \( z_1, z_2, z_3 \) are vertices of a triangle and \( \theta, \phi, \psi \) be the angles opposite to sides \( z_1 - z_2, z_2 - z_3, z_3 - z_1 \) respectively, then (6) yields sine rule of triangles, \( \frac{|z_2 - z_1|}{\sin \theta} = \frac{|z_2 - z_3|}{\sin \phi} = \frac{|z_3 - z_1|}{\sin \psi} = 2r \).

(ix) Euler’s line of an isosceles triangle is perpendicular to the third side. The centre \( c \) as in (1) can also be expressed in terms of chords \( z_2 - z_3 \) and \( z_1 - z_3 \). Then, with notation \( \beta = (z_3 - z_1) (\overline{z_2} - \overline{z_1}) \) and \( \gamma = (z_1 - z_2) (\overline{z_3} - \overline{z_2}) \), we find that

\[
c = \frac{z_1 + z_2 + z_3}{3} + \frac{1}{6i} \left( \frac{\Re \alpha}{\Im \alpha} (z_1 - z_2) + \frac{\Re \beta}{\Im \beta} (z_2 - z_3) + \frac{\Re \gamma}{\Im \gamma} (z_3 - z_1) \right)
\]

(7)

For \( |z_3 - z_1| = |z_3 - z_2|, \beta = \gamma \). So, from (7), \( c - \frac{z_1 + z_2 + z_3}{3} = \frac{1}{6i} \left( \frac{\Re \alpha}{\Im \alpha} - \frac{\Re \beta}{\Im \beta} \right) (z_1 - z_2) \).

This shows that the Euler’s line passing through circumcentre \( c \) and centroid \( \frac{z_1 + z_2 + z_3}{3} \) is perpendicular to the third side \( z_1 - z_2 \).

(x) The circumradius in terms of the lengths of the sides of triangle. We have

\[
|z_2 - z_1|^2 = |z_2 - z_3 + z_3 - z_1|^2 = |z_2 - z_3|^2 + |z_3 - z_1|^2 - 2 \Re (z_2 - z_3) (\overline{z_1} - \overline{z_3})
\]

(8)
Denote $|z_2 - z_3|, |z_3 - z_1|$ and $|z_2 - z_1|$ by $f_1, f_2$ and $f_3$ respectively. Then, from (8),
\[
\text{Re} \alpha = \frac{f_1^2 + f_2^2 - f_3^2}{2} \quad \text{and} \quad (\text{Im} \alpha)^2 = |\alpha|^2 - (\text{Re} \alpha)^2 = f_1^2 f_2^2 - \frac{(f_1^2 + f_2^2 - f_3^2)^2}{4}. \tag{9}
\]
Therefore, from (2),
\[
r^2 = \frac{f_1^2 f_2^2 f_3^2}{4 f_1^2 f_2^2 - (f_1^2 + f_2^2 - f_3^2)^2}. \tag{10}
\]

(i) Distance between centre and chord. From (4) and (10),
\[
|c - \frac{z_1 + z_2}{2}|^2 = r^2 - \frac{f_3^2}{4} = 1 + \frac{f_3^2}{4} \frac{(f_1^2 + f_2^2 - f_3^2)^2}{4 f_1^2 f_2^2 - (f_1^2 + f_2^2 - f_3^2)^2}. \tag{11}
\]

(ii) The distance between the circumcentre and centroid of a triangle. We note that $\text{Im} \alpha = \text{Im} \beta = \text{Im} \gamma$ and therefore, from (7), with notation $\bar{z} = \frac{z_1 + z_2 + z_3}{3},$
\[
|c - \bar{z}|^2 = \frac{1}{36 (\text{Im} \alpha)^2} \left( (\text{Re} \alpha)^2 f_3^2 + (\text{Re} \beta)^2 f_1^2 + (\text{Re} \gamma)^2 f_2^2 - 6 \text{Re} \alpha \text{Re} \beta \text{Re} \gamma \right). \tag{11}
\]
On using (9), we find from (11) that
\[
|c - \bar{z}|^2 = \frac{f_1^2 f_2^2 f_3^2}{4 f_1^2 f_2^2 - (f_1^2 + f_2^2 - f_3^2)^2} - \frac{f_1^2 + f_2^2 + f_3^2}{9}. \tag{11}
\]

4 Conclusion

The arithmetical and geometrical properties of complex numbers provide alternative ways to look at the related geometrical properties. We here demonstrated this in the special case when the geometrical properties of a circle follow from its equation in the complex form.

References
