Ideals of the Algebra

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Abstract
We construct an algebra $A$ such that $A$ has a nonempty finite set $\Delta$ of associative and commutative binary operations. Then we may define an ideal with respect to a nonempty subset of $\Delta$. If some hypotheses are satisfied, then we have that a union of the ideals is an ideal. An ideal $M$ is maximal with respect to a subset of $\Delta$ if there is not an ideal $J \neq A$ such that $J$ contains $M$. And an algebra is local with respect to a subset of $\Delta$ if it has a unique maximal ideal. Suppose that the algebra $A$ is local with respect to $\Phi$ and $\Psi$, $M$ and $N$ are the maximal ideals, respectively, and $J$ is an ideal with respect to $\Phi \cup \Psi$. Then we have that $J \subseteq M \cap N$ if some conditions hold. Let $A$ be a local algebra with respect to $\Phi$, $M$ the maximal ideal. For all $\Psi$ with $\Phi \subseteq \Psi \subseteq \Delta$, if $M$ is an ideal with respect to $\Psi$, then $A$ is local with respect to $\Psi$. A preimage of an ideal with respect to $\Phi$ under a homomorphism is an ideal with respect to $\Phi$.

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1. INTRODUCTION

Let $\Delta := \{\beta_1, \ldots, \beta_n\}$ be a set of binary operation symbols. Suppose that the ordered pair $\mathcal{A} := (\Delta, \sigma)$ is an algebraic language such that $\mathcal{A}$ contains binary operations which are commutative and associative, and suppose that $A$ is an algebra of the language $\mathcal{A}$, see notation 3.1 and convention 3.1 for the details.

We may define an ideal with respect to a nonempty subset of $\Delta$ in an algebra $A$, see definition 3.1 and examples 3.1 and 3.2 for more details.

If subalgebras $I$ and $J$ are ideals with respect to $\Phi \subseteq \Delta$ and $\Psi \subseteq \Delta$ in $A$, respectively, then the subset $I \cup J$ is a subalgebra, see propositions 3.1 and 3.3 and corollary 3.1.1 for more details. The subalgebra $I \cup J$ is an ideal if the hypotheses of propositions 3.2 and 3.4 and corollary 3.2.1 are satisfied.

In definition 3.2, we define a maximal ideal with respect to a subset of $\Delta$ in $A$. Let $M$ be a maximal ideal with respect to $\Phi$. We have that if $M$ is an ideal with respect
to $\Psi$ then $M$ is maximal with respect to $\Psi$ for all $\Psi$ with $\Phi \subseteq \Psi$, see proposition 3.5 for more details.

And if an algebra $A$ contains a unique maximal ideal with respect to a subset of $\Delta$, then the algebra $A$ is called local, see definition 3.3 for the details.

Suppose that an algebra $A$ is local with respect to $\Phi \subseteq \Psi$ and $\Phi \subseteq \Psi$, see proposition 3.6 and corollary 3.6.1 for more details. Thus if $J = M$, then the algebra $A$ is local with respect to $\Psi$. This is discussed in corollary 3.6.2.

Suppose that an algebra $A$ is local with respect to $\Phi \subseteq \Psi$ and $\Phi \subseteq \Psi$, $\Phi \subseteq \Psi$, and $M$ and $N$ are the maximal ideals, respectively, and $M \cap N \neq \emptyset$. If $J \subseteq A$ is an ideal with respect to $\Phi \subseteq \Psi$ and $\Phi \subseteq \Psi$, then $J = M$ implies that the algebra $A$ is local with respect to $\Phi \subseteq \Psi$, see corollaries 3.6.3 to 3.6.5 for more details.

Let $f : A \rightarrow B$ be a homomorphism of algebras of the language $\mathfrak{L}$. We have that $f$ makes the subalgebra $f^{-1}(J)$ to be an ideal if $J$ is an ideal with respect to $\Phi$ in $B$, see proposition 3.7 for more details.

2. Preliminaries

Recall some definitions in universal algebra.

**Definition 2.1 ([4, 5]).** An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) language provided that

- $L$ is a nonempty set,
- $\sigma : L \rightarrow \mathbb{N}$ is a mapping.

A language $\langle L, \sigma \rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then $f$ is called an operation symbol, and $\sigma(f)$ is called the arity of $f$. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then $r$ is called a relation symbol, and $-\sigma(r)$ is called the arity of $r$. A language is said to be algebraic if it has no relation symbols.

**Definition 2.2 ([4]).** Let $X$ be a nonempty class and $n$ a nonnegative integer. Then an $n$-ary partial operation on $X$ is a mapping from a subclass of $X^n$ to $X$. If the domain of the mapping is $X^n$, then it is called an $n$-ary operation. And an $n$-ary relation is a subclass of $X^n$ where $n > 0$. An operation(relation) is said to be unary, binary or ternary if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called nullary if the arity is 0.

**Definition 2.3 ([4]).** An ordered pair $A := \langle A, \mathfrak{L} \rangle$ is said to be a structure of a language $\mathfrak{L}$ if $A$ is a nonempty class and there exists a mapping which assigns to every $n$-ary operation symbol $f \in \mathfrak{L}$ an $n$-ary operation $f^A$ on $A$ and assigns to every $n$-ary relation symbol $r \in \mathfrak{L}$ an $n$-ary relation $r^A$ on $A$. If all operation on $A$ are partial operations, then $A$ is called a partial structure. A (partial)structure $A$ is said to be a (partial)algebra if the language $\mathfrak{L}$ is algebraic.
Definition 2.4 ([4, 5]). Let $A, B$ be (partial) structures of a language $\mathcal{L}$. A mapping $\varphi: A \rightarrow B$ is said to be a homomorphism provided that

$$
\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n)) \text{ for every } n\text{-ary operation } f;
$$

$$
r^A(a_1, \ldots, a_n) \implies r^B(\varphi(a_1), \ldots, \varphi(a_n)) \text{ for every } n\text{-ary relation } r.
$$

Definition 2.5 (cf. [4, 5]). Let $X$ be a nonempty set. Suppose that $\beta$ is a binary operation on $X$. Then the 2-ary operation $\beta$ is associative provided that

$$
\beta(a, \beta(b, c)) = \beta(\beta(a, b), c) \text{ for every } a, b, c \in X.
$$

Definition 2.6 (cf. [4, 5]). With the notations of definition 2.5, the 2-ary operation $\beta$ is commutative provided that

$$
\beta(a, b) = \beta(b, a) \text{ for every } a, b \in X.
$$

3. Ideals of the Algebras

Convention 3.1. We assume that all binary operations are associative [definition 2.5] and commutative [definition 2.6] in this paper.

Notation 3.1. Let $\Delta := \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a set of operation symbols for $n > 0$, and $\sigma: \Delta \rightarrow \mathbb{Z}$ a map which assigns to $\beta_i$ for all $\beta_i \in \Delta$. Then the ordered pair $\mathcal{L} := \langle \Delta, \sigma \rangle$ is an algebraic language [definition 2.1]. It is clear that all operations of the language $\mathcal{L}$ are binary operations. Suppose that $A$ is an algebra [definition 2.3] of the language $\mathcal{L}$.

Definition 3.1. Let the notations be as in notation 3.1, and $\Phi \subseteq \Delta$ a nonempty subset of 2-ary operations on $A$. A nonempty subalgebra $J$ is said to be an ideal with respect to $\Phi$ provided that $\beta_i \in \Phi$ implies $\beta_i^J(a, x) \in J$ for all $a \in J, x \in A$. In this case, we say that the nonempty subset $\Phi \subseteq \Delta$ makes the subalgebra $J$ to be an ideal.

Remark 3.1. We have an immediate consequence of definition 3.1. For all nonempty subset $\Psi \subset \Phi$, if $J$ is an ideal with respect to $\Phi$, then $J$ is an ideal with respect to $\Psi$. And the converse need not hold.

Example 3.1 (cf. [1–3]). Let $\mathcal{L} := \{+\}, 0, 1\}$ where the map $\sigma$ is given by assigning $2$ to $+$ and $. Then a commutative ring $R$ is an algebra of the language $\mathcal{L}' := \{+,\}$.

Hence an ideal in the ring $R$ is an ideal with respect to $\{\}$ in $R$ (as an algebra of the language $\mathcal{L}'$).

Example 3.2 (cf. [4, 5]). Let $B := \{B, v, \wedge, '0, 1\}$ be a boolean algebra. Hence the boolean algebra $B$ can be regarded as an algebra of the language $\mathcal{B} := \{v, \wedge, \}$.

Then an ideal in $B$ is an ideal with respect to $\{\}$ in $B$ (as an algebra of the language $\mathcal{B}$), and a filter in $B$ is an ideal with respect to $\{v\}$.

Proposition 3.1. Let the notations be as in notation 3.1, $\beta_i \neq \beta_j \in \Delta$. Suppose that subalgebras $I$ and $J$ are ideals with respect to $\Delta \setminus \{\beta_i\}$ and $\Delta \setminus \{\beta_j\}$ in $A$, respectively. Then the subset $I \cup J$ is a subalgebra of $A$.
Proof. It suffices to prove that \( \beta(x,y) \in I \cup J \) for \( \beta \in \Delta, x \in I, y \in J \), since \( I \) and \( J \) are subalgebras. For every \( x \in I, y \in J \), we have that
\[
\beta_k(x,y) = \begin{cases} 
I & \text{if } \beta_k = \beta_i, \\
J & \text{if } \beta_k = \beta_j, \\
I \cap J & \text{otherwise.}
\end{cases}
\]
Observe that \( I \cap J \) is not empty. Therefore, the subset \( I \cup J \) is a subalgebra. □

Remark 3.2. Let the notations be as in notation 3.1, \( I \) and \( J \) ideals with respect to \( \Phi \) and \( \Psi \), respectively. Then we have that \( \Phi \cap \Psi \neq \emptyset \) implies \( I \cap J \neq \emptyset \), since we have that \( \beta(x,y) \in I \cap J \), for all \( x \in I, y \in J \), and all \( \beta \in \Phi \cap \Psi \).

Corollary 3.1.1. Let the notations be as in notation 3.1, and \( \Phi, \Psi \subset \Delta \) with \( \Psi \cap \Phi = \emptyset \). If subalgebras \( I \) and \( J \) are ideals with respect to \( \Delta \setminus \Phi \) and \( \Delta \setminus \Psi \) in \( A \), respectively, then the subset \( I \cup J \) is a subalgebra.

Proof. Obviously. □

Proposition 3.2. With the same hypotheses as in proposition 3.1, if \( \{\beta_i, \beta_j\} \neq \Delta \) then the subalgebra \( I \cup J \) is an ideal with respect to \( \Delta \setminus \{\beta_i, \beta_j\} \).

Proof. By remark 3.1, we have that \( I \) and \( J \) are ideals with respect to \( \Delta \setminus \{\beta_i, \beta_j\} \), since \( \{\beta_i\}^C \cap \{\beta_j\}^C = (\{\beta_i\} \cup \{\beta_j\})^C \). Hence we have that \( \beta_k \in \Delta \setminus \{\beta_i, \beta_j\} \) implies \( \beta_k(x,y) \in I \cup J \), for every \( x \in I, y \in J \), since we have \( \beta(x,y) \in I \cap J \). It follows that the subalgebra \( I \cup J \) is an ideal with respect to \( \Delta \setminus \{\beta_i, \beta_j\} \). □

Corollary 3.2.1. With the hypotheses of corollary 3.1.1, if \( \Phi \cup \Psi \neq \Delta \) then the subalgebra \( I \cup J \) is an ideal with respect to \( \Delta \setminus (\Phi \cup \Psi) \).

Proof. Obviously. □

The two following propositions are just restatements of corollaries 3.1.1 and 3.2.1, respectively.

Proposition 3.3. Let the notations be as in notation 3.1, \( I \) and \( J \) ideals with respect to \( \Phi \subset \Delta \) and \( \Psi \subset \Delta \) in \( A \), respectively. We have that the subset \( I \cup J \) is a subalgebra of \( A \) if \( \Phi \cup \Psi = \Delta \).

Proof. Let \( \beta \in \Delta, x, y \in I \cup J \). Since \( I \) and \( J \) are subalgebras, \( \Phi \cup \Psi = \Delta \). It suffices to show that \( \beta(x,y) \in I \cup J \) for all \( x \in I, y \in J, \beta \in \Delta \). By definition 3.1, we have that \( \beta \in \Phi \) or \( \beta \in \Psi \) implies \( \beta(x,y) \in I \) or \( \beta(x,y) \in J \), respectively, for all \( x \in I, y \in J \). It follows that \( I \cup J \) is a subalgebra. □

Proposition 3.4. Let the notations be as in proposition 3.3. If \( \Phi \cup \Psi = \Delta \) and \( \Psi \cap \Phi \neq \emptyset \), then the subalgebra \( I \cup J \) is an ideal with respect to \( \Psi \cap \Phi \).

Proof. It is clear that \( \beta(x, a) \in I \cup J \) for every \( x \in A, a \in I \cup J \), and every \( \beta \in \Psi \cap \Phi \). Hence the proposition is an immediate consequence of definition 3.1. □
Remark 3.3. Let the notations be as in proposition 3.3. It is clear that the subset \( \Psi \cap \Phi \) makes the subalgebra \( I \cap J \) to be an ideal if \( \Phi \cap \Psi \neq \emptyset \). And the subalgebra \( I \cap J \) need not be an ideal with respect to \( \Phi \cup \Psi \) if \( \Phi \neq \Psi \), since there may be \( x \in A, y \in I \cap J \) such that \( \beta(x, y) \in I \) but \( \beta(x, y) \notin I \cap J \) for some \( \beta \in \Phi \setminus \Psi \).

Definition 3.2 (cf. \([1, 4, 5]\)). Let the notations be as in notation 3.1, and \( \Phi \subseteq \Delta \). An ideal \( M \) with respect to \( \Phi \) in \( A \) is said to be maximal if \( M \neq A \) and for every ideal \( N \) with respect to \( \Phi \) such that \( M \subset N \subset A \), either \( M = N \) or \( N = A \).

Remark 3.4. Let the notations be as in notation 3.1, \( M \) a maximal ideal with respect to \( \Phi \). The ideal \( M \) need not be maximal with respect to \( \Psi \) for \( \Psi \subseteq \Phi \).

Proposition 3.5. Let the notations be as in notation 3.1. Suppose that \( M \) is a maximal ideal with respect to \( \Phi \) of the algebra \( A \). For all \( \Psi \) with \( \Phi \subseteq \Psi \), we have that if \( M \) is an ideal with respect to \( \Psi \) then \( M \) is maximal with respect to \( \Psi \). And there is no an ideal \( N \neq A \) with respect to \( \Psi \) such that \( M \subset N \), i.e., if \( \Psi \) makes \( N \neq A \) to be an ideal, then we have \( M \subset N \), for all \( \Psi \) with \( \Phi \subseteq \Psi \).

Proof. We assume that \( N \neq A \) is a maximal ideal with respect to \( \Psi \), and \( M \subseteq N \). By remark 3.1, we have that \( N \) is an ideal with respect to \( \Phi \). This is a contradiction. Hence we have \( M = N \) or \( M \not\subseteq N \). Therefore, the proposition holds. \( \square \)

Definition 3.3 (cf. \([1–3]\)). Let the notations be as in notation 3.1, and \( \Phi \subseteq \Delta \). The algebra \( A \) is local with respect to \( \Phi \) provided that \( A \) has a unique maximal ideal with respect to \( \Phi \).

Proposition 3.6. Let the notations be as in notation 3.1, and \( \beta_i \in \Delta \). Suppose that \( A \) is local with respect to \( \{\beta_i\} \), and \( M \) is the maximal ideal. For all \( \beta_j \in \Delta \), if \( J \not\subseteq A \) is an ideal with respect to \( \{\beta_i, \beta_j\} \) then \( J \subseteq M \).

Proof. Observe remark 3.1, we have that the subset \( \{\beta_i\} \subset \{\beta_i, \beta_j\} \) makes \( J \) to be an ideal. Therefore, we have \( J \subseteq M \). \( \square \)

Corollary 3.6.1. Let the notations be as in notation 3.1, and \( \Phi \subseteq \Delta \). Suppose that \( A \) is local with respect to \( \Phi \), and \( M \) is the maximal ideal. For all subset \( \Psi \subseteq \Delta \) with \( \Phi \subseteq \Psi \), if \( J \not\subseteq A \) is an ideal with respect to \( \Psi \), then \( J \subseteq M \).

Proof. Obviously. \( \square \)

Corollary 3.6.2. Let the notations be as in notation 3.1. Suppose that \( A \) is local with respect to \( \Phi \), and \( M \) is the maximal ideal. For all \( \Psi \subseteq \Psi \subset \Delta \), we have that if \( M \) is an ideal with respect to \( \Psi \) then the algebra \( A \) is local with respect to \( \Psi \) and \( M \) is the unique maximal ideal.

Proof. This is an immediate consequence of proposition 3.5 and corollary 3.6.1. \( \square \)

Remark 3.5. For \( \Psi \subseteq \Phi \), the algebra \( A \) defined in corollary 3.6.2 need not be local with respect to \( \Psi \), since the ideal \( M \) need not be unique maximal with respect to \( \Psi \), cf. remarks 3.1 and 3.4.
Corollary 3.6.3. Let the notations be as in notation 3.1, $\Phi \not= \Psi \subset \Delta$. Suppose that $A$ is local with respect to $\Phi$ and $\Psi$, $M$ and $N$ are the maximal ideals, respectively, and $M \cap N \not= \emptyset$. If $J \not= A$ is an ideal with respect to $\Theta$ then $J \subset M \cap N$, for all $\Theta$ with $\Phi \cup \Psi \subset \Theta \subset \Delta$.

Proof. By corollary 3.6.1, we have $J \subset M$ and $J \subset N$. It follows that $J \subset M \cap N$. □

Remark 3.6. The subalgebra $M \cap N$ need not be an ideal, cf. remark 3.3. But we have the two following corollaries which are consequences of corollaries 3.6.2 and 3.6.3.

Corollary 3.6.4. With the hypotheses of corollary 3.6.3, for all $\Theta$ with $\Phi \cup \Psi \subset \Theta \subset \Delta$, if $\Theta$ makes $M \cap N$ to be an ideal, then the algebra $A$ is local with respect to $\Theta$.

Proof. For all ideal $J$ with respect to $\Theta$, we have that $J \subset M \cap N$ by corollary 3.6.3. This suffices to prove that if $\Theta$ makes $M \cap N$ to be an ideal, then $M \cap N$ is a unique maximal ideal with respect to $\Theta$ by corollary 3.6.2. Thus the algebra $A$ is local with respect to $\Theta$. □

Corollary 3.6.5. With the hypotheses of corollary 3.6.3, for all $\Theta$ with $\Phi \cup \Psi \subset \Theta \subset \Delta$, if the subset $M \cap N = \emptyset$, then there is no an ideal $J$ with respect to $\Theta$ such that $J \not= A$.

Proof. Obviously. □

Proposition 3.7. Let the notations be as in notation 3.1, $B$ an algebra of the language $\mathfrak{A}$, and $J$ an ideal with respect to $\Phi \subset \Delta$ in $B$. Suppose that $f: A \rightarrow B$ is a homomorphism[definition 2.4]. We have that the inverse image $f^{-1}(J)$ is an ideal with respect to $\Phi$.

Proof. Let $I := f^{-1}(J)$. It is clear that $I$ is a subalgebra of $A$. It suffices to prove that $\beta^A(a,x) \in I$. For all $a \in I, x \in A$ and all $\beta \in \Phi$, we have that $\beta^B(f(a), f(x)) \in J$ implies that $\beta^A(a,x) \in I$, since we have $\beta^B(f(a), f(x)) = f(\beta^A(a,x))$. This completes the proof. □