A (1.999999)-approximation ratio for vertex cover problem

Majid Zohrehbandian
Department of Mathematics, Karaj Branch,
Islamic Azad University, Karaj, Iran. zohrebandian@yahoo.com

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Abstract

The vertex cover problem is a famous combinatorial problem, and its complexity has been heavily studied. While a 2-approximation can be trivially obtained for it, researchers have not been able to approximate it better than 2-$o(1)$. In this paper, by introducing a new semidefinite programming formulation that satisfies new properties, we introduce an approximation algorithm for the vertex cover problem with a performance ratio of 1.999999 on arbitrary graphs, en route to answering an open question about the unique games conjecture.

Keywords: Combinatorial Optimization, Vertex Cover Problem, Unique Games Conjecture, Complexity Theory.

1 Introduction

In complexity theory, the abbreviation *NP* refers to "nondeterministic polynomial", where a problem is in *NP* if we can quickly (in polynomial time) test whether a solution is correct. *P* and *NP*-complete problems are subsets of *NP* problems. We can solve *P* problems in polynomial time while determining whether or not it is possible to solve *NP*-complete problems quickly (called the *P vs NP* problem) is one of the principal unsolved problems in Mathematics and Computer science.

Here, we consider the vertex cover problem (VCP), a famous *NP*-complete problem, which cannot be approximated within a factor of 1.36 [1], unless $P =$

NP. In contrast, a 2-approximation factor can be trivially obtained by taking all the vertices of a maximal matching in the graph. However, improving this simple 2-approximation algorithm is hard [2, 3].

In this paper, we propose a new semidefinite programming (SDP) formulation, and based on its solution, we introduce a VCP feasible solution with an approximation ratio of 1.999999 on arbitrary graphs.

The rest of the paper is structured as follows. Section 2 is about the vertex cover problem and introduces new conditions on its solution. In section 3, using a new SDP model whose solution satisfies the introduced conditions, we propose a solution algorithm for VCP with a performance ratio of 1.999999 on arbitrary graphs. Finally, Section 4 concludes the paper.

2 Performance ratio based on VCP feasible solutions

In the mathematical discipline of graph theory, a vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The problem of finding a minimum vertex cover is a typical example of an *NP*-complete optimization problem. In this section, we examine the ratio of an arbitrary VCP feasible solution to its optimal value, in both the presence and absence of a lower bound for the VCP optimal value. Then, in the next section, we introduce a 1.999999-approximation ratio for the vertex cover problem on arbitrary graphs.

Let $G = (V, E)$ be an undirected graph on vertex set V and edge set E, where |V|$= n$. Throughout this paper, $z^*(G)$ is the optimal value for the vertex cover problem on G, and VCP feasible solutions have been introduced by a vertex partitioning $V = V_1 \cup V_0$ with an objective value $\mid V_1 \mid$. The integer linear programming (ILP) model for VCP is as follows; i.e. $z1^* = z^*(G)$.

$$(1)\ min_{s.t.}\ z1 = \sum_{i \in V} x_i$$

$$x_i + x_j \geq 1 \quad ij \in E$$

$$x_i \in \{0, +1\} \quad i \in V$$

Lemma 1. **[4]** Let x^* be an extreme optimal solution to the linear programming (LP) relaxation of the model (1). Then $x^*_j \in \{0, 0.5, 1\}$ for $j \in V$. If we define $V^0 = \{j \in V \mid x^*_j = 0\}$, $V^{0.5} = \{j \in V \mid x^*_j = 0.5\}$ and $V^1 = \{j \in V \mid x^*_j = 1\}$, then there exists a VCP optimal solution which includes all of the vertices V^1, and it is a subset of $V^{0.5} \cup V^1$.

Lemma 2. Let x^* be an extreme optimal solution to the LP relaxation of the model (1), and $V^0 = \{j \in V \mid x_j^* = 0\}$, $V^{0.5} = \{j \in V \mid x_j^* = 0.5\}$, $V^1 = \{j \in V \mid x_j^* = 1\}$, and $G_{0.5}$ be the induced graph on the vertices $V^{0.5}$. If we can introduce a vertex cover feasible partitioning $V^{0.5} = V_1^{0.5} \cup V_0^{0.5}$ with an approximation ratio of $1 \leq \rho < 2$, for the VCP on $G_{0.5}$, then the vertex cover feasible partitioning $V = (V_1 \cup V_0) = (V_1^{0.5} \cup V^1) \cup (V_0^{0.5} \cup V^0)$, has an approximation ratio of $1 \leq \rho < 2$, for the VCP on G.
Proof. Based on the approximation ratio of $\frac{|V_1^{0.5}|}{z^*(G_{0.5})} \leq \rho$, we have,

$$| V_1^{0.5} | + | V^1 | \leq \rho z^*(G_{0.5}) + \rho \mid V^1 \mid$$

Therefore, $\frac{|V_1|}{z^*(G)} = \frac{|V_1^{0.5}|+|V^1|}{z^*(G_{0.5})+|V^1|} \leq \rho \diamond$

Based on the Lemma (2), it is sufficient to produce an approximation ratio of $1 \leq \rho < 2$, on $G_{0.5}$. Moreover, $z^*(G_{0.5}) \geq \frac{|V^{0.5}|}{2}$. Then, it is sufficient to focus on graphs G where $z^*(G) \geq \frac{n}{2}$.

We know that we can efficiently solve the following SDP formulation, as a relaxation of the VCP model (1).

$$(2)\ min_{s.t.}\ z2 = \sum_{i \in V} X_{oi}$$

$$X_{oi} + X_{oj} - X_{ij} = 1 \quad ij \in E$$

$$X_{ii} = 1, \quad 0 \leq X_{ij} \leq +1 \quad i, j \in V \cup \{o\}$$

$$X \succeq 0$$

Moreover, by introducing the unit vectors $v_o, v_1, ..., v_n$, the SDP model (2) can be written as follows, where $v_i^T v_j = X_{ij}$, and $V_1 = \{i \in V \mid v_i = v_o\}$ is a feasible vertex cover, and $V_o = V - V_1$ is the set of orthogonal vectors to v_o.

$$(3)\ min_{s.t.}\ z3 = \sum_{i \in V} v_o^T v_i$$

$$v_o^T v_i + v_o^T v_j - v_i^T v_j = 1 \quad ij \in E$$

$$v_i^T v_i = 1, \quad 0 \leq v_i^T v_j \leq +1 \quad i, j \in V \cup \{o\}$$

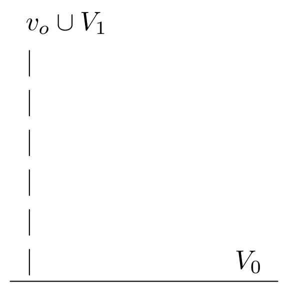

Figure 1. A VCP feasible solution

Lemma 3. Let $z^*(G) \geq \frac{n}{2} + \frac{n}{k} = \frac{(k+2)n}{2k}$ be a lower bound on VCP optimal value. Then, all vertex cover feasible partitioning $V = V_1 \cup V_0$ satisfy the approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+2} < 2$.
Proof. If $z^*(G) \geq \frac{(k+2)n}{2k}$, then $\frac{n}{z^*(G)} \leq \frac{2k}{k+2}$. Therefore,

$$\frac{\mid V_1 \mid}{z^*(G)} \leq \frac{n}{z^*(G)} \leq \frac{2k}{k+2} < 2$$

and this completes the Proof $\diamond$

Lemma 4. Let $z^*(G) \geq \frac{n}{2}$, and $V = V_1 \cup V_0$ is a VCP feasible partitioning, where $\mid V_1 \mid \leq \frac{kn}{k+1}$ and $\mid V_0 \mid \geq \frac{n}{k+1}$ (or $\mid V_1 \mid \leq k \mid V_0 \mid$). Based on such a solution, we have an approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+1} < 2$.
Proof. If $\mid V_1 \mid \leq \frac{kn}{k+1}$, then $n \geq \frac{k+1}{k} \mid V_1 \mid$. Hence, $z^*(G) \geq \frac{n}{2} \geq \frac{k+1}{2k} \mid V_1 \mid$ and $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+1} < 2$ $\diamond$

In the next section, we will introduce a new SDP model to find a suitable lower bound or feasible solution and apply Lemma (3) or Lemma (4).

3 A (1.999999)-approximation algorithm on arbitrary graphs

To find a suitable lower bound or feasible solution, we introduce the following conditions concerning the optimal solution of the SDP model (3).

Condition a) $\mid \{j \in V : v_o^{*T} v_j^* < 0.5\} \mid < 0.000001n$.
Condition b) $\mid \{j \in V : v_o^{*T} v_j^* > 0.5004\} \mid < 0.01n$.

Lemma 5. If $z^*(G) \geq \frac{n}{2}$ and the optimal solution of the SDP model (3) does not satisfy Conditions (a) and (b), then, based on such a solution, we can produce a VCP feasible solution with a performance ratio of 1.999999.

Proof. If the optimal solution of the SDP model (3) does not satisfy Condition (a), then we can introduce $V_0 = \{j \in V \mid v_o^{*T} v_j^* < 0.5\}$ and $V_1 = V - V_0$, to have a VCP feasible solution with $\mid V_0 \mid \geq 0.000001n$ and $\mid V_1 \mid \leq 0.999999n \leq 999999 \mid V_0 \mid$. Therefore, for such a solution and based on the Lemma (4), we have an approximation ratio of $\frac{|V_1|}{z^*(G)} < \frac{2(999999)}{999999+1} = 1.999998 < 1.999999$.

Otherwise, if the optimal solution of the SDP model (3) satisfies Condition (a) but does not satisfy Condition (b), then there exists the following lower bound on $z^*(G)$ value.

$$z^*(G) \geq z3^* \geq (0)(0.000001n)_{\{s.t.\ v_o^{*T} v_j^* < 0.5\}}$$

$$+(0.5)(0.989999n)_{\{s.t.\ v_o^{*T} v_j^* \geq 0.5\}} + (0.5004)(0.01n)_{\{s.t.\ v_o^{*T} v_j^* > 0.5004\}}$$

$$= \frac{n}{2} + 0.0000035n$$

Note that, Condition (a) has been satisfied and as a result, we have less than 0.000001n of vertices $j \in V$ with $v_o^{*T} v_j^* < 0.5$. Condition (b) has not been satisfied and we have more than 0.01n of vertices $j \in V$ with $v_o^{*T} v_j^* > 0.5004$. Therefore, in the above inequality, the first summation is the lower bound on the vertices $j \in V$ with $v_o^{*T} v_j^* < 0.5$, and the third summation is the lower bound on only 0.01n of the vertices $j \in V$ with $v_o^{*T} v_j^* > 0.5004$, and beyond the 0.01n of such vertices are considered in the second summation. Moreover, the second summation is the lower bound on the other vertices (the vertices $j \in V$ with $0.5 \leq v_o^* v_j^* \leq 0.5004$, and the vertices $j \in V$ with $v_o^{*T} v_j^* > 0.5004$ beyond the 0.01n of such vertices considered in the third summation).

Therefore, based on the above lower bound on $z^*(G)$ value and the Lemma (3), all VCP feasible solutions $V = V_1 \cup V_0$ satisfy the approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2(\frac{1}{0.0000035})}{\frac{1}{0.0000035}+2} < 1.999999$ $\diamond$

Definition 1. Let ε=0.0004, and $V_\varepsilon = \{j \in V \mid 0.5 \leq v_o^{*T} v_j^* \leq 0.5 + \varepsilon\}$, and $E_\varepsilon = \{ij \in E \mid i, j \in V_\varepsilon\}$.

Note that, after solving the SDP model (3) on problems with $z^*(G) \geq \frac{n}{2}$,
- If the solution does not satisfy Conditions (a) and (b), we can produce a VCP feasible solution with a ratio factor 1.999999.

- Otherwise, $| V_\varepsilon |\geq 0.989999n$, and $0 \leq v_i^{*T} v_j^* \leq 2\varepsilon \ \ (ij \in E_\varepsilon)$; i.e. the corresponding vectors of the edges in E_ε are almost orthogonal to each other.

Therefore, to produce a VCP performance ratio of 1.999999 for problems with $z^*(G) \geq \frac{n}{2}$, we need a solution for the SDP model (3) that does not satisfy Conditions (a) or (b). To do this, we will introduce a new SDP model, whose optimal solution does not satisfy Conditions (a) and (b) unless the induced graph on V_ε is bipartite. In other words, we want to approximately satisfy the conditions of the following theorem while still being able to use the result of the theorem.

Theorem 1. For any positive integer t, there does not exist $2t+1$ unit vectors $v_1, ..., v_{2t+1}$ such that $v_j^T v_{j+1} = 0$, and $v_j + v_{j+1} = v_{2t+1} + v_1$ $(j = 1, ..., 2t)$, and $v_{2t+1}^T v_1 = 0$.
Proof. Let $u = v_{2t+1} + v_1$. Then $| u |= \sqrt{2}$, and $u^T v_j = 1$ $(j = 1, ..., 2t+1)$. Moreover,

$$(v_1 + v_2) + (v_2 + v_3) + ... + (v_{2t} + v_{2t+1}) + (v_{2t+1} + v_1) = (2t+1)U$$

Hence,

$$2(v_1 + ... + v_{2t} + v_{2t+1}) = (2t+1)u$$

and

$$(v_1 + v_2) + (v_3 + v_4) + ... + (v_{2t-1} + v_{2t}) + v_{2t+1} = (t + 0.5)u$$

Therefore, $tu + v_{2t+1} = (t + 0.5)u$, and this concludes that $v_{2t+1} = 0.5u$ which is a contradiction $\diamond$

Clime 1. For any positive integer t, there does not exist $2t+1$ unit vectors $v_1, ..., v_{2t+1}$ such that all consecutive vectors v_j, v_{j+1} $(j = 1, ..., 2t)$, and v_{2t+1}, v_1 are almost orthogonal to each other, and $v_j + v_{j+1}$ $(j = 1, ..., 2t)$ and $v_{2t+1} + v_1$ are almost equal to a vector u with $| u |= \sqrt{2}$, and $u^T v_j = 1$ $(j = 1, ..., 2t+1)$.
Proof. Read the rest $\diamond$

Let $G2 = (V_{new}, E_{new})$ be a new graph, where we add two adjacent vertices a and b to the graph G, and connect all vertices of G to them; i.e. $V_{new} = V \cup \{v_a, v_b\}$. Then, based on the SDP model (3), and by introducing the unit vectors $v_o, v_1, ..., v_n, v_a, v_b$, we introduce a new SDP model as follows, where

$V_{1new} = V_1 = \{i \in V_{new} \mid v_i = v_o\}$ corresponds to a feasible vertex cover on graph G, and $V_{0new} = V_0 = V - V_1$ corresponds to orthogonal vectors to v_o.

$$(4)\ min_{s.t.}\ z4 = \sum_{i \in V} v_o^T v_i$$

$$SDP\ (3)\ constraints\ on\ G$$

$$v_o^T v_i + v_o^T v_j - v_i^T v_j = 1 \quad i \in V,\ j \in \{a, b\}$$

$$-0.5 \leq v_i^T v_j \leq +0.5 \quad i \in V,\ j \in \{a, b\}$$

$$v_i^T v_i = 1, \quad v_o^T v_i = \ +0.5 \quad i \in \{a, b\}$$

$$v_a^T v_b = \ 0$$

Lemma 6. Due to the additional constraints, we have $z4^* \geq z3^*$. Moreover, to produce a feasible solution for the SDP model (4) on $G2$, we can add suitable vectors v_a and v_b to each VCP feasible partitioning $V = V_1 \cup V_0$ on G, where $v_i^T v_j = \ +0.5$ for $i \in V_1,\ j \in \{a, b\}$, and $v_i^T v_j = \ -0.5$ for $i \in V_0,\ j \in \{a, b\}$ (For example, for $v_o = v_i = [0.5, 0.5, 0.5, 0.5]^t \in V_1$ and $v_i = [-0.5, -0.5, 0.5, 0.5]^t \in V_0$, we can introduce $v_a = e_1 = [1, 0, 0, 0]^t$, and $v_b = e_2 = [0, 1, 0, 0]^t$). Therefore, $z4^* \leq z^*(G)$.

We are going to prove that, there does not exist an optimal SDP (4) solution such that satisfies Conditions (a) and (b) on G, unless the induced graph on V_ε is bipartite.

Theorem 2. For four unit vectors v_1, v_2, v_3, v_4 which are orthogonal to each other, there exists exactly one unit vector v with $v^T v_i = 0.5$ $(i = 1, 2, 3, 4)$. Such a vector v satisfies the equation $v = 0.5(v_1 + v_2 + v_3 + v_4)$.

Proof.

Due to $v_1^T v_2 = 0$, we have $\mid v_1 + v_2 \mid = \sqrt{\mid v_1 \mid^2 + \mid v_2 \mid^2} = \sqrt{2}$.
Due to $v_3^T v_4 = 0$, we have $\mid v_3 + v_4 \mid = \sqrt{\mid v_3 \mid^2 + \mid v_4 \mid^2} = \sqrt{2}$.
Due to $(v_1 + v_2)^T (v_3 + v_4) = 0$, we have

$$\mid v_1 + v_2 + v_3 + v_4 \mid = \sqrt{\mid v_1 + v_2 \mid^2 + \mid v_3 + v_4 \mid^2} = 2$$

Moreover, we have $(v_1 + v_2 + v_3 + v_4)^T v = 2$. Therefore,

$$\mid v_1 + v_2 + v_3 + v_4 \mid\mid v \mid cos(\theta) = 2$$

and this concludes that $\theta = 0$ and $v = 0.5(v_1 + v_2 + v_3 + v_4)$ $\diamond$

proposition 1. Let $w = 0.5(v_1+v_2+v_3+v_4)$, for four unit vectors v_1, v_2, v_3, v_4 which are almost orthogonal to each other. Then, a unit vector v with $0.5 \leq v^T v_i \leq 0.5 + \varepsilon$ $(i = 1, 2, 3, 4)$ is almost equal to w. In other words, there exists a vector r with $w + r = v$, where $-\varepsilon \leq \mid w \mid - \mid v \mid \leq \varepsilon$, $\mid r \mid \leq \varepsilon$, and $cos(v, w) \geq 1 - \varepsilon$.

Theorem 3. Let $\theta(v, w) = cos^{-1}(\frac{v^T w}{|v||w|})$. For $n+1$ vectors $v_o, v_1, ..., v_n$ with $0^o \leq \theta(v_i, v_j) \leq 90^o$ $(i, j = o, 1, ..., n)$, we have

$$\theta(v_o, \sum_{i=1}^{n} v_i) \leq max\{\theta(v_o, v_i) : \ i = 1, ..., n\}$$

In other words, the angle between the vectors v_o and $w = \sum_{i=1}^{n} v_i$ is smaller than the maximum angle between the pair of vectors v_o and v_i $(i = 1, ..., n)$.

Proof. We give proof by induction on n. For $n = 3$ vectors v_o, v_1, v_2, if $\theta(v_o, v_1 + v_2) > max\{\theta(v_o, v_1), \theta(v_o, v_2)\}$ then $\theta(v_o, v_1 + v_2) > \theta(v_o, v_1)$ and $\theta(v_o, v_1 + v_2) > \theta(v_o, v_2)$. Hence, $cos(v_o, v_1 + v_2) < cos(v_o, v_1)$ and $cos(v_o, v_1 + v_2) < cos(v_o, v_2)$. Therefore,

$$\frac{v_o^T(v_1 + v_2)}{\mid v_1 + v_2 \mid} < \frac{v_o^T v_1}{\mid v_1 \mid} \quad and \quad \frac{v_o^T(v_1 + v_2)}{\mid v_1 + v_2 \mid} < \frac{v_o^T v_2}{\mid v_2 \mid}$$

which conclude

$$2(\frac{v_o^T(v_1 + v_2)}{\mid v_1 + v_2 \mid}) < \frac{v_o^T v_1 \mid v_2 \mid + v_o^T v_2 \mid v_1 \mid}{\mid v_1 \mid\mid v_2 \mid}$$

and

$$2v_o^T v_1 \mid v_1 \mid\mid v_2 \mid + 2v_o^T v_2 \mid v_1 \mid\mid v_2 \mid <$$

$$v_o^T v_1 \mid v_2 \mid\mid v_1 + v_2 \mid + v_o^T v_2 \mid v_1 \mid\mid v_1 + v_2 \mid$$

and

$$v_o^T v_1 \mid v_2 \mid (2 \mid v_1 \mid - \mid v_1 + v_2 \mid) + v_o^T v_2 \mid v_1 \mid (2 \mid v_2 \mid - \mid v_1 + v_2 \mid) < 0$$

However,

$$0 > v_o^T v_1 \mid v_2 \mid (2 \mid v_1 \mid - \mid v_1 + v_2 \mid) + v_o^T v_2 \mid v_1 \mid (2 \mid v_2 \mid - \mid v_1 + v_2 \mid)$$

$$\geq (min\{v_o^T v_1 \mid v_2 \mid, v_o^T v_2 \mid v_1 \mid\}) \times 2(\mid v_1 \mid + \mid v_2 \mid - \mid v_1 + v_2 \mid) \geq 0$$

which is a contradiction. Therefore, it is true for $n = 3$. Suppose it is true for $n = k - 1$, and we want to prove it for $n = k$. For $t < k$, our inductive hypothesis implies that

$$\theta(v_o, w_1 = \sum_{i=1}^{t} v_i) \leq max\{\theta(v_o, v_i) : \ i = 1, ..., t\}$$

$$\theta(v_o, w_2 = \sum_{i=t+1}^{k} v_i) \leq max\{\theta(v_o, v_i) : \ i = t+1, ..., k\}$$

Therefore,

$$\theta(v_o, \sum_{i=1}^{k} v_i) \leq max\{\theta(v_o, w_i) : \ i = 1, 2\} \leq max\{\theta(v_o, v_i) : \ i = 1, ..., k\}$$

and this completes the proof $\diamond$

Lemma 7. Based on the optimal solution of the SDP model (4), and by introducing $u = 2v_o^* - v_a^* - v_b^*$, we have

$$| \ u \ |= \sqrt{2}, \quad and \quad \forall j \in V_\varepsilon : \ u^T v_j^* = 1$$

Proof.

$$| \ u \ |= \sqrt{u^T u} = \sqrt{4 - 1 - 1 - 1 + 1 + 0 - 1 + 0 + 1} = \sqrt{2}$$

Moreover, for each vertex j in V_ε, we have

$$v_o^{*T} v_c^* + v_o^{*T} v_j^* - v_c^{*T} v_j^* = 1 \quad c \in \{a, b\}, \quad j \in V_\varepsilon$$

Therefore, we obtain

$$v_c^{*T} v_j^* = -0.5 + v_o^{*T} v_j^* \quad c \in \{a, b\}, \quad j \in V_\varepsilon$$

which concludes that $u^T v_j^* = 2v_o^{*T} v_j^* - v_a^{*T} v_j^* - v_b^{*T} v_j^* = 2v_o^{*T} v_j^* + 0.5 - v_o^{*T} v_j^* + 0.5 - v_o^{*T} v_j^* = 1$, and the angle between two vectors u and v_j^* is 45 degrees for $j \in V_\varepsilon$ $\diamond$

Now we can prove our main result.

Theorem 4. There does not exist an optimal solution for the SDP model (4) on $G2$ such that satisfies Conditions (a) and (b) on G, unless the induced graph on V_ε is bipartite.

Proof. Suppose that the optimal solution of the SDP model (4) satisfies Conditions (a) and (b) on G. Therefore, for each edge $ij \in E_\varepsilon$, unit vectors $v_i^*, v_j^*, v_a^*, v_b^*$ are almost orthogonal to each other, and the unit vector v_o^* is almost equal to $0.5(v_i^* + v_j^* + v_a^* + v_b^*)$, where $0.5 \leq v_o^{*T} v_l \leq 0.5 + \varepsilon$ $(l = i, j, a, b)$. In other words, $v_i^* + v_j^*$ is almost equal to $u = 2v_o^* - v_a^* - v_b^*$ and there exists a vector $r_{i,j}$ with $v_i^* + v_j^* + r_{i,j} = u$, where $-\varepsilon \leq \mid v_i^* + v_j^* \mid - \mid u \mid \leq \varepsilon$, $\mid r_{i,j} \mid \leq \varepsilon$, and $cos(u, v_i^* + v_j^*) \geq 1 - \varepsilon$.

For any positive integer t, if we have an odd cycle on $2t + 1$ vertices in $G_\varepsilon = (V_\varepsilon, E_\varepsilon)$, then, by addition of the vectors in this cycle and introducing $w = (v_1 + v_2) + (v_2 + v_3) + ... + (v_{2t} + v_{2t+1}) + (v_{2t+1} + v_1)$, we have

$$u^T w = \ 2(2t + 1) = \mid u \mid\mid w \mid cos(u, w)$$

Where, $\theta(u, w) \leq max\{\theta(u, v_1 + v_2), ..., \theta(u, v_{2t} + v_{2t+1}), \theta(u, v_{2t+1} + v_1)\} \leq cos^{-1}(1 - \varepsilon) \leq 1^o$.

By introducing $w' = \sum_{i=1}^{2t+1} v_i = \ 0.5w$, the above summation can be written as follows,

$$u^T w = 2u^T w' = 2 \mid u \mid\mid w' \mid cos(u, w') = \ 2(2t + 1)$$

Where, $\theta(u, w') = \theta(u, w) \leq cos^{-1}(1 - \varepsilon) \leq 1^o$, and

$$\frac{(t + 0.5)\sqrt{2}}{1} \leq \mid w' \mid = \frac{(t + 0.5)\sqrt{2}}{cos(u, w')}$$

By introducing $w_i = w' - v_i$, for $i = 1, ..., 2t + 1$, the following properties are satisfied:

$$\mid v_i \mid^2 = \mid w' \mid^2 + \mid w_i \mid^2 - 2w'^T w_i \qquad (I)$$

$$\mid w' \mid \leq \mid w_i \mid + \mid v_i \mid = \mid w_i \mid + 1 \qquad (II)$$

$$\sum_{i=1}^{2t+1} w_i = (2t + 1)w' - \sum_{i=1}^{2t+1} v_i = 2tw' \qquad (III)$$

$$u^T w_i = \ 2t = \mid u \mid\mid w_i \mid cos(u, w_i) \qquad (IV)$$

Where, $\theta(u, w_i) \leq max\{\theta(u, v_{i+1} + v_{i+2}), ..., \theta(u, v_{i-2} + v_{i-1})\} \leq cos^{-1}(1 - \varepsilon) \leq 1^o$, and

$$\frac{t\sqrt{2}}{1} \leq \mid w_i \mid = \frac{t\sqrt{2}}{cos(u, w_i)}$$

By addition of the equation (I) in this cycle, we have

$$\sum_{i=1}^{2t+1} \mid v_i \mid^2 = \sum_{i=1}^{2t+1} \mid w' \mid^2 + \sum_{i=1}^{2t+1} \mid w_i \mid^2 - 2 \sum_{i=1}^{2t+1} w'^T w_i$$

Hence,

$$\begin{aligned} 2t+1 &= (2t+1) \mid w' \mid^2 + \sum_{i=1}^{2t+1} \mid w_i \mid^2 - 2w'^T \sum_{i=1}^{2t+1} w_i \\ &= (2t+1) \mid w' \mid^2 + \sum_{i=1}^{2t+1} \mid w_i \mid^2 - 2w'^T (2tw') \\ &= (-2t+1) \mid w' \mid^2 + \sum_{i=1}^{2t+1} \mid w_i \mid^2 \qquad (V) \end{aligned}$$

Let $\mid w_j \mid = max\{\mid w_i \mid : \; i = 1, ..., 2t+1\}$. Then, based on the right side of the equation (V) we have

$$(-2t+1) \mid w' \mid^2 + \sum_{i=1}^{2t+1} \mid w_i \mid^2$$

$$\begin{aligned} &\leq (-2t+1)(\mid w_j \mid + 1)^2 + (2t+1) \mid w_j \mid^2 \\ &= 2 \mid w_j \mid^2 - 4t \mid w_j \mid + 2 \mid w_j \mid - 2t + 1 \\ &= \frac{4t^2}{cos^2(u, w_j)} - \frac{4\sqrt{2}t^2}{cos(u, w_j)} + \frac{2\sqrt{2}t}{cos(u, w_j)} - 2t + 1 \\ &\leq \frac{4t^2}{cos^2(1^o)} - \frac{4\sqrt{2}t^2}{cos(0^o)} + \frac{2\sqrt{2}t}{cos(1^o)} - 2t + 1 \\ &\leq -1.6556t^2 + 0.8288t + 1 \end{aligned}$$

However, $-1.6556t^2 + 0.8288t + 1$ is less than $2t+1$, and this contradicts the equation (V).

Therefore, there does not exist an odd cycle in G_ε, and G_ε is bipartite $\diamond$

proposition 2. To produce a performance ratio of 1.999999 for problems with $z^*_{VCP} \geq \frac{n}{2}$, we should solve the SDP model (4) on $G2$, and if the solution satisfies Conditions (a) and (b), we should solve the VCP problem on the bipartite graph G_ε, where $\mid V_\varepsilon \mid \geq 0.989999n$.

Moreover, based on the lemma (2) and the Proposition (2), to produce a performance ratio of 1.999999 for problems with $z^*_{VCP} < \frac{n}{2}$, it is sufficient to produce an extreme optimal solution for the LP relaxation of the model (1) and introducing $G2$ based on $G_{0.5}$.

Theorem 5. The Optimal solution of the following LP model corresponds to an extreme optimal solution of the LP relaxation of the model (1).

$$(5)\ min_{s.t.} z5 = \sum_{i=1}^{n} (0.1)^i x_i$$

$$x_i + x_j \geq 1 \quad ij \in E$$

$$\sum_{i \in V} x_i = z^*_{LP\ relaxation\ of\ the\ model\ (1)}$$

$$0 \leq x_i \leq +1 \quad i \in V$$

Proof. The feasible region of the model (5) is an optimal face of the feasible region of the LP relaxation of the model (1), and its optimal solution corresponds to the solution of the following algorithm, based on the priority weights of the decision variables.

Step 0. Let z^* be the optimal value of the LP relaxation of the model (1), and k=1.

Step k. Solve the following LP model.

$$(6)\ min_{s.t.} z(k) = x_k$$

$$x_i + x_j \geq 1 \quad ij \in E$$

$$\sum_{i \in V} x_i = z^*$$

$$x_i = x_i^* = z(k)^* \quad i = 1, \cdots, k-1$$

$$0 \leq x_i \leq +1 \quad i \in V$$

Let k=k+1. If $k < n$ repeat this step, otherwise, the solution x^* is an extreme optimal solution of the LP relaxation of the model (1) $\diamond$

Therefore, our algorithm to produce an approximation ratio of 1.999999, for arbitrary vertex cover problems, is as follows:

Mahdis Algorithm (To produce a vertex cover solution on graph G with a ratio factor $\rho \leq 1.999999$)

Step 1. Let z^* be the optimal value of the LP relaxation of the model (1) on G, and $V^1 = V^0 = \{\}$

Step 2. If $z^* \geq \frac{n}{2}$, then go to Step 3. Otherwise, solve the LP model (5) to produce an extreme optimal solution of the LP relaxation of the model (1), and based on the solution ($x_j^* \in \{0, 0.5, 1\}\ \ j \in V$), introduce $V^0 = \{j \in V \mid x_j^* = 0\}$, $V^{0.5} = \{j \in V \mid x_j^* = 0.5\}$, $V^1 = \{j \in V \mid x_j^* = 1\}$, and let $G = G_{0.5}$ as the induced graph on the vertex set $V^{0.5}$.

Step 3. Produce $G2$ based on G and solve the SDP (4) model.

Step 4. If $\mid \{j \in V :\ v_o^{*T} v_j^* < 0.5\} \mid > 0.000001n$ (the solution does not satisfy Condition (a)), then introduce $V_0 = \{j \in V \mid v_o^{*T} v_j^* < 0.5\}$ and $V_1 = V - V_0$ to produce a suitable solution $V_1 \cup V_0$ which satisfies $\frac{|V_1|}{z^*(G)} \leq 1.999999$. Then, go to Step 7. Otherwise, go to Step 5.

Step 5. If $\mid \{j \in V :\ v_o^{*T} v_j^* > 0.5004\} \mid > 0.01n$, then it is sufficient to produce an arbitrary VCP feasible solution $V = V_1 \cup V_0$ to have $\frac{|V_1|}{z^*(G)} \leq 1.999999$ and go to Step 7. Otherwise, go to Step 6.

Step 6. The solution satisfies Conditions (a) and (b). Hence, based on the Theorem (4), graph G_ε is bipartite, and $\mid V_\varepsilon \mid \geq 0.989999n$. Therefore, solve the VCP problem on bipartite subgraph G_ε and add all vertices of $V - V_\varepsilon$ to the solution to produce a feasible solution $V_1 \cup V_0$ which satisfies $\frac{|V_1|}{z^*(G)} \leq 1.999999$. Then, go to Step 7.

Step 7. The partitioning $(V_1 \cup V^1) \cup (V_0 \cup V^0)$ produces a VCP feasible solution on the original graph G with an approximation ratio factor $\rho \leq 1.999999$.

proposition 3. Based on the proposed 1.999999-approximation algorithm for the vertex cover problem, the unique games conjecture is not true.

4 Conclusions

One of the open problems regarding the vertex cover problem is the possibility of introducing an approximation algorithm within any constant factor better than 2. Here, we propose a new algorithm to produce a 1.999999-approximation ratio for the vertex cover problem on arbitrary graphs, which leads to the conclusion that the unique games conjecture is not true.

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Competing Interest and Data Availability

The authors have no relevant financial or non-financial interests to declare relevant to this article's content. Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

References

1) Dinur I., Safra S. (2005) On the hardness of approximating minimum vertex cover. Annals of Mathematics, 162, 439-485.
2) Khot S. (2002) On the power of unique 2-Prover 1-Round games. Proceeding of 34th ACM Symposium on Theory of Computing, STOC.
3) Khot S., Regev O. (2008) Vertex cover might be hard to approximate to within $2-\varepsilon$. Journal of Computer and System Sciences, 74, 335-349.
4) Nemhauser G.L., Trotter L.E. (1975) Vertex packing: Structural properties and algorithms. Mathematical Programming, 8, 232-248.