The Time-Neutral Exterior Schwarzschild Metric in Polynomial Form

Kevin Loch

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The time-neutral metric is introduced and the time-neutral exterior Schwarzschild metric is converted to polynomial form in \( r \) and total mass \( M \). The polynomials are found to be cubic with no constant term, which allows the two non-zero roots of each to be extracted from the reduced quadratic form.

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THE TIME-NEUTRAL METRIC

A time-neutral metric is an exact solution to the Einstein field equations\(^1\) multiplied by the inverse length scalar \( \frac{1}{c^2 dt^2} \), and set equal to one. The time-neutral metric with signature \((+, -, -, -)\) is given by:

\[
\frac{ds^2}{c^2 dt^2} = \frac{dτ^2}{dt^2} = 1.
\] (1)

Multiplying the line element by this scalar is a special coordinate transformation that also converts coordinate infinitessimals to dimensionless ratios\(^2\).

Setting the transformed metric to one represents neutral time dilation \((\Delta t' = \Delta t)\), as opposed to maximal time dilation of the zero-time metric \( \frac{dτ^2}{dt^2} = 0 \). That finite, real, solutions to time-neutral metrics exist with \( M \neq 0 \) may be surprising. It is especially counter-intuitive for the Schwarzschild metric\(^2\), which does not have the opposite sign \( r^2 Q \) or \( a^2 \) factors that give additional degrees of freedom\(^3\) to higher order metrics. However, these solutions do exist mathematically.

Further transforming the time-neutral metric into polynomial form has the effect of replacing the divergence at \( r = 0 \) in the standard form of the metric, with the constraint \( v^2 r / c^2 \neq 0 \) in the polynomial form. Even if this is not physical it might be a useful tool for quantum gravity research.

The open-source software tool \texttt{knsolver}\(^5\) can be used to generate plots of solutions to the time-neutral and zero-time metrics. It uses numerical methods on the standard form of the Kerr-Newman\(^6\) metric which is flexible but slow, especially for precision results. Converting a time-neutral metric to polynomial form may allow for fast exact solutions, depending on the degree of the polynomial.

The time-neutral metric should not be confused with metrics of neutral signature\(^7\)–\(^11\) \((+, +, -, -)\), which are often referred to as “neutral metrics” in mathematical literature.

POLYNOMIAL FORM

As shown in the proof below, the time-neutral exterior Schwarzschild metric in spherical coordinates \((t, r, \theta, \varphi)\), with test particle velocities converted to \( v^2 / c^2 \), as a polynomial in \( r \) and total mass \( M \) are given by:

\[
\left(\frac{v^2}{c^2} + \frac{v^2_\theta}{c^2}\right) r s r^3 + \left(1 - \frac{v^2_\theta}{c^2}\right) r_s^2 r^2 - r_s^3 r = 0,
\] (2)

\[
\frac{8G^3 r}{c^6} M^3 - \left(1 - \frac{v^2_\theta}{c^2}\right) \frac{4G^2 r^2}{c^4} M^2 - \left(\frac{v^2}{c^2} + \frac{v^2_\theta}{c^2}\right) \frac{2G r^3}{c^2} M = 0,
\] (3)

\[
r_s = \frac{2GM}{c^2}, \quad (r \geq R), \quad (v^2 / c^2 \neq 0),
\] (4)

\(^*\) kevin@loch.me

\(^1\) In metrics where angular momentum \( J \neq 0 \), an arbitrary finite non-zero value can be assigned to the remaining linear \( dt \) factors without affecting the neutral metric. \texttt{knsolver} sets this to \( t_P \), but other values such as 1, or even \(-1\) also work. The remaining linear \( d\varphi \) factors must then be derived from \( d\varphi = v_\varphi dt / r \).
As both are cubic with no constant term, the two non-zero roots of each can be recovered from the reduced quadratic equations:

\[
\left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right)r_s v^2 + \left(1 - \frac{v_\Omega^2}{c^2}\right)r_s^2 r - r_s^3 = 0,
\]

\[
\frac{8G^3 r}{c^5} M^2 - \left(1 - \frac{v_\Omega^2}{c^2}\right)\frac{4G^2 r^2}{c^4} M - \left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right)\frac{2Gr^3}{c^2} = 0,
\]

\[
r_s = \frac{2GM}{c^2}, \quad (r \geq R), \quad \left(\frac{v_r^2}{c^2} \neq 0\right).
\]

This agrees with the two non-zero roots for each implied by plots generated from knsolver when electric charge \(Q\) and angular momentum \(J\) are set to zero.

**PROOF**

Deriving the polynomials involves basic algebraic manipulation but special care must be taken to not lose the degrees of freedom represented by the inverse sum \((1 - \frac{v_\Omega^2}{c^2})^{-1}\) in the \(dr^2\) term. This proof also uses some substitutions that are not strictly necessary here but can be used to help manage the proliferation of length factors in the higher order metrics.\(^6\) \(^7\) \(^8\)

We begin with the exterior \((r \geq R)\) Schwarzschild metric\(^2\) in spherical coordinates \((t, r, \theta, \varphi)\) and metric signature \((+, -, -, -)\):

\[
ds^2 = c^2 dt^2 = \left(1 - \frac{r_s}{r}\right)c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (r \geq R),
\]

\[
r_s = \frac{2GM}{c^2}, \quad d\Omega^2 = d\theta^2 + \sin \theta d\varphi^2.
\]

Divide the metric by \(c^2 dt^2\) and set equal to one to obtain the time-neutral metric, also convert test particle velocities to \(v^2/c^2\). This is the equation we will convert to polynomial form:

\[
\frac{dr^2}{dt^2} = \left(1 - \frac{r_s}{r}\right) - \frac{v_r^2}{c^2} \left(1 - \frac{r_s}{r}\right)^{-1} - \frac{v_\Omega^2}{c^2} = 1, \quad (r \geq R),
\]

\[
\frac{v_r^2}{c^2} = \frac{dr^2}{c^2 dt^2}, \quad \frac{v_\Omega^2}{c^2} = \frac{r^2 d\Omega^2}{c^2 dt^2}.
\]

Multiply the \(dt^2\) and \(d\Omega^2\) terms by \(\left(1 - \frac{v_r^2}{c^2}\right)\) \(/\left(1 - \frac{v_\Omega^2}{c^2}\right)\), then simplify the denominator by dividing both sides by \(r\). At this point we must add the constraint \(v_r^2/c^2 dt^2 \neq 0\) as we are combining the \(dr^2\) term inverse length factor with the other terms in the denominator.

\[
(r \geq R), \quad \left(\frac{v_r^2}{c^2} \neq 0\right),
\]

\[
\frac{\left(1 - \frac{r_s}{r}\right) \left(1 - \frac{r_s}{r}\right) - \frac{v_r^2}{c^2} \left(1 - \frac{r_s}{r}\right)}{1 - \frac{r_s}{r}} = 1,
\]

\[
\frac{\left(1 - \frac{r_s}{r}\right) \left(1 - \frac{r_s}{r}\right) - \frac{v_r^2}{c^2} \left(1 - \frac{r_s}{r}\right)}{r - r_s} = \frac{1}{r}.
\]

Multiply out the \(dt^2\) term product, then simplify the numerator by multiplying both sides by \(r^2\), then group by terms of \(r\) being especially careful with signs of combined factors:

\[
\frac{1 - \frac{2r_s}{r} + \frac{r_s^2}{r^2} - \frac{v_r^2}{c^2} \left(1 - \frac{r_s}{r}\right)}{r - r_s} = \frac{1}{r}.
\]
\[
\frac{r^2 - 2r_s r + r_s^2 - \frac{v^2}{c^2} r^2 - \frac{v_s^2}{c^2} (r^2 - r_s r)}{r - r_s} = r,
\]

(17)

\[
\frac{\left(1 - \frac{v^2}{c^2} - \frac{v_s^2}{c^2}\right) r^2 - \left(2 - \frac{v^2}{c^2}\right) r_s r + r_s^2}{r - r_s} = r.
\]

(18)

At this point we introduce temporary variables \(X, Y\), to help capture the extra degrees of freedom in the inverse sum. Let \(X\) be the numerator on the left side of equation (18):

\[
\frac{X}{r - r_s} = r,
\]

(19)

\[
X = \left(1 - \frac{v^2}{c^2} - \frac{v_s^2}{c^2}\right) r^2 - \left(2 - \frac{v^2}{c^2}\right) r_s r + r_s^2.
\]

(20)

The denominator sum is now split between numerators \(X\) and \(Y\):

\[
\frac{X}{r - r_s} = \frac{X}{r} + \frac{Y}{r_s},
\]

(21)

\[
\frac{X}{r - r_s} = \frac{X r_s + Y r}{r_s r}.
\]

(22)

It is now safe to multiply both sides by \((r - r_s) r_s r\), then we group by terms of \(X\) and \(Y\):

\[
X r_s r = (X r_s + Y r) (r - r_s),
\]

(23)

\[
X r_s r = X r_s r - X r_s^2 + Y r^2 - Y r_s r,
\]

(24)

\[
X r_s^2 = Y \left(r^2 - r_s r\right).
\]

(25)

Substitute symbols \(X_f, Y_f\), for length factors associated with \(X\) and \(Y\), then solve for \(Y\):

\[
XX_f = YY_f,
\]

(26)

\[
X_f = r_s^2, \quad Y_f = r^2 - r_s r,
\]

(27)

\[
Y = \frac{XX_f}{Y_f}.
\]

(28)

With equations (23) and (22), multiply by \(r_s r\) and substitute for \(Y\) using equation (28)

\[
\frac{X r_s + Y r}{r_s r} = r,
\]

(29)

\[
X r_s + \frac{XX_f r}{Y_f} = r_s r^2.
\]

(30)

Multiply by \(Y_f\), then group by terms of \(X\):

\[
XY_f r_s + XX_f r = Y_f r_s r^2,
\]

(31)

\[
X (Y_f r_s + X_f r) - Y_f r_s r^2 = 0.
\]

(32)
Substitute symbols $F_1, F_2, F_3$, for the combined length factors:

$$X(F_1 + F_2) - F_3 = 0,$$

(33)

$$F_1 = Y_r r_s, \quad F_2 = X_f r, \quad F_3 = Y_f r s^2.$$  

(34)

With equations 22 and 23, solve $F_1 + F_2$, and with equation 24, solve $X(F_1 + F_2)$:

$$F_1 = r_s r^2 - r_s^2 r, \quad F_2 = r_s^2 r,$$

(35)

$$F_1 + F_2 = r_s r^2,$$

(36)

$$X(F_1 + F_2) = \left[1 - \frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right] r^2 - \left(2 - \frac{v_0^2}{c^2}\right) r_s r + r_s^2,$$

(37)

$$X(F_1 + F_2) = \left(1 - \frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right) r_s r^4 - \left(2 - \frac{v_0^2}{c^2}\right) r_s r^3 + r_s^3 r^2.$$

(38)

Solve $F_3$ and $X(F_1 + F_2) - F_3$:

$$F_3 = r_s r^4 - r_s^2 r^3,$$

(39)

$$X(F_1 + F_2) - F_3 = \left(1 - \frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right) r_s r^4 - \left(2 - \frac{v_0^2}{c^2}\right) r_s^2 r^3 + r_s^3 r^2 - r_s r^4 + r_s^2 r^3 = 0,$$

(40)

With equations 25 and 26, divide by $r$ as the smallest degree of $r$ is 2:

$$\left(1 - \frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right) r_s r^4 - \left(2 - \frac{v_0^2}{c^2}\right) r_s^2 r^3 + r_s^3 r^2 - r_s r^4 + r_s^2 r^3 = 0,$$

(41)

$$\left(1 - \frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right) r_s r^3 - \left(2 - \frac{v_0^2}{c^2}\right) r_s^2 r^2 + r_s^3 r - r_s r^3 + r_s^2 r^2 = 0.$$

(42)

Convert to standard polynomial form in $r$ by grouping terms by $r$, then inverting the sign of each term, and applying the constraints from equation 27:

$$\left(-\frac{v^2}{c^2} - \frac{v_0^2}{c^2}\right) r_s r^3 - \left(1 - \frac{v_0^2}{c^2}\right) r_s^2 r^2 + r_s^3 r = 0,$$

(43)

$$\left(\frac{v^2}{c^2} + \frac{v_0^2}{c^2}\right) r_s r^3 + \left(1 - \frac{v_0^2}{c^2}\right) r_s^2 r^2 - r_s^3 r = 0, \quad (r \geq R, \ v_r^2/c^2 \neq 0)$$

(44)

Convert to standard polynomial form in $M$ by substituting $r_s = \frac{2GM}{c^2}$, then grouping terms by $M$ and inverting signs again:

$$\left(\frac{v^2}{c^2} + \frac{v_0^2}{c^2}\right) \frac{2GM}{c^2} r^3 + \left(1 - \frac{v_0^2}{c^2}\right) \frac{4G^2M^2}{c^4} r^2 - \frac{8G^3M^3}{c^6} r = 0,$$

(45)

$$\frac{8G^3r}{c^6} M^3 - \left(1 - \frac{v_0^2}{c^2}\right) \frac{4G^2v_r^2}{c^4} M^2 - \left(\frac{v^2}{c^2} + \frac{v_0^2}{c^2}\right) \frac{2Gr^3}{c^3} M = 0, \quad (r \geq R, \ v_r^2/c^2 \neq 0)$$

(46)


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