Energy-momentum relation in curved space-time

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Abstract

In this paper we will obtain a variation of the energy-momentum relation:

$$E^2 = m^2c^4 + p^2c^2$$

With an element more that depends on the Ricci scalar curvature:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

This leads to a variation in the definition the energy of a mass at rest:

$$E = mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2c^2}}$$

In small gravitational fields, the term $\frac{R\hbar^2}{m^2c^2}$ is so small that the equation is indistinguishable from the classical equation:

$$E = mc^2$$

But it can have its importance in high gravitational fields as a limit to the curvature is imposed:

$$R < \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2}$$

Also, we will get the variation Klein-Gordon equation in flat space-time:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2} - R \right)\psi(t, \boldsymbol{x}) = 0$$

And in curved space time:

$$\frac{-1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right) + \left(\frac{m^2c^2}{\hbar^2} - R\right)\psi = 0$$

The same way, we will obtain a variation to the Dirac equation, both in flat and curved space time:

$$i\gamma^\mu\partial_\mu\psi = \sqrt{\frac{m^2c^2}{\hbar^2} - R\psi}$$

$$i\gamma^\mu D_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R\psi}$$

The last point studied is how the obtained equation:

$$E^2 = m^2 c^4 + p^2 c^2 - R\hbar^2 c^2$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Has an element (R, representing the Ricci scalar curvature) that is acting reducing the energy of the particles. It represents a reduction of the Energy but in units Length[-2] in the latter equation. This value of R is calculated (considering different metrics for the universe -interior and exterior Schwarzschild metric-) to find that it corresponds almost exactly with the expected value of the cosmological constant (the effects created by the Dark Energy):

$$R_{intSchw} = 1.603E - 52m^{-2}$$

$$R_{intSchw1/3} = 0.5345E - 52m^{-2}$$

$$R_{extSchw} \rightarrow \sqrt{K} = 1.852E - 52m^{-2}$$

While the cosmological constant is in the order of:

$$\Lambda = 1.1056E - 52m^{-2}$$

So, making this R a perfect candidate for the Dark Energy. But not as an added element to match the observations, on the contrary, an element that appears directly in the equations resulting that it matches the observations.

Keywords

1. Introduction

In this paper will use the previous paper [73] to explain the consequences of the equations obtained in that paper.

2. Energy-momentum relation

In [73] we obtained the following factor in the Klein-Gordon equation of the part of the Energy of the particle (being the Ricci scalar curvature):

$$\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)$$

This is:

$$\frac{m^2 c^2}{\hbar^2} - R$$

We see that the mass is squared, typical for equations for squared energy as in the squared Energy equation by Einstein [75] (and also in Klein-Gordon equation[67]). So, we keep this mass squared, but we can multiply by constants (changing the units but not the meaning of the elements). Multiplying now by $\hbar^2 c^2$ to get units of energy squared in both elements.

$$m^2 c^4 - R\hbar^2 c^2$$

So, this is the element to be introduced in the squared equation of energy (energy-momentum relation) by Einstein [75]:

$$E^2 = m^2 c^4 + p^2 c^2$$

Leading to:

$$E^2 = m^2 c^4 - R\hbar^2 c^2 + p^2 c^2$$

$$E^2 = m^2 c^4 + p^2 c^2 - R\hbar^2 c^2$$

Being this squared energy equation including also the gravitational effects in the Energy (as one element is the Ricci scalar representing the effects of the gravitational field).

As we saw in [73] the element depending in the Ricci scalar is in general several orders of magnitude smaller than the one depending on the mass. Making it neglectable in general except in high gravitational fields as we will comment later.

If we consider a mass at rest $(p = 0)$ and in a gravitational field, we will have:

$$E^2 = m^2 c^4 - R\hbar^2 c^2$$

Taking the square root:

$$E = \sqrt{m^2 c^4 - R\hbar^2 c^2}$$

$$E = mc^2 \sqrt{1 - \frac{R\hbar^2 c^2}{m^2 c^4}} = mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}}$$

Considering a small gravitational field, we can approximate with two elements of the Taylor series as:

$$E = mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}} \cong mc^2 \left(1 - \frac{1}{2}\frac{R\hbar^2}{m^2 c^2}\right) \cong mc^2 - \frac{1}{2}\frac{\hbar^2}{m}R$$

In the first versions of (73) I did not consider this equation as quadratic and that is the reason, I calculated there wrongly this equation (without the square root or the ½ factor).

We can see that the energy of the particle is reduced by an element that depends on the Ricci scalar and inversely to the mass. Making the total energy of a particle smaller, as the gravitational field where it stands, is smaller.

This reduction is in general negligible [73], being several orders of magnitude below the normal energy. Anyhow, as the mass increases, the Ricci scalar increases also due to gravitational effects. As the Ricci scalar is being subtracted to the energy depending on the mass, the system will arrive to a balance before becoming a singularity.

We can see this, returning to the former squared energy equation:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

$$E = \sqrt{m^2c^4 + p^2c^2 - R\hbar^2c^2}$$

If we consider that energy cannot be imaginary, then:

$$m^2c^4 + p^2c^2 - R\hbar^2c^2 > 0$$

$$m^2c^4 + p^2c^2 > R\hbar^2c^2$$

$$\frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} > R$$

$$R < \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2}$$

We see that the curvature created by the gravitational field has a limit that depends on the mass and momentum of the masses creating is. This is, the probabilities to arrive to a singularity clearly reduces. The higher the mass and momentum, the higher the curvature of the filed, is clear. But also, the higher the curvature the "power" of the energy and the momentum to increase the gravitational field starts reducing.

This means, in the end we arrive to a limit, that is the above equation. The curvature that can be created is limited by the mass and momentum that ironically have originally created it. The term $-R\hbar^2c^2$ in the energy-momentum relation acts somehow as dark energy [78] acting in opposition to the mass related energy.

In fact, going even further talking about the nature of the elemental particles. If we consider the particle as an area of space that arrives to an equilibrium, we could consider that the masses are the discrete values (a kind of eigenvalues) where this equilibrium between the energy of the particle and the gravitational field created (represented by the Ricci scalar) arrives to a balance.

This could be one of the reasons that there are three generations or families of masses per type of particle [76]. Because, as we saw in [73] in this context, the three special dimensions are the ones to be taken into account, and therefore the tensors and matrices have indices going from 1 to 3 (three row or columns in the case of matrices with 3 eigenvalues). The reason of not considering time as a 4th dimension, is because time corresponds to one of the 8 degrees of freedom created by these spatial dimensions (considered normally the trivector created by the three spatial dimensions or sometimes the scalars depending on the situation see Annex A1, [5][47] [63] [73].

3. Klein-Gordon equation

In our Energy-momentum relation equation:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

Let's divide by $\hbar^2c^2$:

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

The Klein-Gordon equation in Euclidean metric according to literature is [67]:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t,\boldsymbol{x}) = 0$$

The energy and the momentum are represented by the derivatives through tome and space respectively. And the mass element has its own element. We see that what is missing is the element regarding the Ricci Scalar. With the units that the equation is in now (L^{-2}) can include it directly in the equation (the units of R are also L^{-2}). See two equations above to check that.

So, the result is:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2} - R\right)\psi(t,\boldsymbol{x}) = 0$$

If you have seen my papers, you will know, that this equation is squared so it should be applied to the wavefunction collapsed (squared) not to the single function (but in standard algebra to do this is not possible). It is possible to do it in geometric algebra but this is another story that you can check in Annex A1.

According to literature the Klein Gordon equation [67] in curved spacetime is:

$$\frac{-1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right) + \frac{m^2c^2}{\hbar^2}\psi = 0$$

Adding the new element:

$$\frac{-1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right) + \frac{m^2c^2}{\hbar^2}\psi - R\psi = 0$$

$$\frac{-1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right) + \left(\frac{m^2c^2}{\hbar^2} - R\right)\psi = 0$$

In geometric Algebra the one obtained is [73]:

$$e^\beta\nabla_\beta(\nabla_\alpha(\psi^\dagger\psi)e^\alpha) = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi$$

$$e^\beta\nabla_\beta(\nabla_\alpha(\psi^\dagger\psi)e^\alpha) = \left(\frac{m^2c^2}{\hbar^2} - R\right)\psi^\dagger\psi$$

Where, as commented, the equation is applied to the wavefunction squared and is symmetric from left to right. Making it completely symmetric and taking the license of considering that ∇_α applies to the left (in [73] you can see a nomenclature for this operation, but here just consider that applies to the left):

$$e^\beta\nabla_\beta\left((\psi^\dagger\psi)\nabla_\alpha e^\alpha\right) = \sqrt{\frac{m^2c^2}{\hbar^2} - R}\,\psi^\dagger\psi\sqrt{\frac{m^2c^2}{\hbar^2} - R}$$

We can get the Dirac equation in geometric algebra:

$$e^\beta\nabla_\beta\psi^\dagger = \sqrt{\frac{m^2c^2}{\hbar^2} - R}\,\psi^\dagger$$

And:

$$(\psi\nabla_\alpha)e^\alpha = \psi\sqrt{\frac{m^2c^2}{\hbar^2} - R}$$

Or putting again ∇_α in the normal order and applying to the right again:

$$(\nabla_\alpha \psi)e^\alpha = \sqrt{\frac{m^2 c^2}{\hbar^2} - R\psi}$$

We can see that they are the same equation but reversed and applying to a function that is reversed compared to the equation in ∇_β. You can see the meaning of reverse and geometric algebra in [73].

One thing we can see here is that the square root, this time does not include the momentum (the momentum is in the operators in the left-hand side). So probably the restriction to R would be even higher, and be:

$$\frac{m^2 c^2}{\hbar^2} - R > 0$$

$$R < \frac{m^2 c^2}{\hbar^2}$$

Where all the considerations regarding the previous chapter (reducing the probabilities of singularities and that the masses of particles could be eigenvalues to an equilibrium limit between created gravitational field created and reduction of mass due to it) keeps being the same. Except that this equation is more restricting and does not include the momentum of the particle just the mass.

4. The Dirac equation

The Dirac equation in flat spacetime according to literature [13] is:

$$\left(i\hbar\gamma^\mu \partial_\mu - mc\right)\psi(x) = 0$$

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0$$

$$i\hbar\gamma^\mu \partial_\mu \psi = mc\psi$$

$$i\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar}\psi$$

As we know that the R has to be subtracted to the square of the element, we can do the following:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2}}\psi$$

And now we can adapt it, just subtracting that element regarding R to the equation:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R\psi}$$

We can check in (74) that in curved space time, the equation in the literature is similar but applying covariant derivative instead of the derivative. So, we can follow, the same exact steps to obtain the equation:

$$i\hbar\gamma^\mu D_\mu \psi - mc\psi = 0$$

$$i\hbar\gamma^\mu D_\mu \psi = mc\psi$$

$$i\gamma^\mu D_\mu \psi = \frac{mc}{\hbar}\psi$$

$$i\gamma^\mu D_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2}}\,\psi$$

Now we subtract the element R to the equation:

$$i\gamma^\mu D_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\,\psi$$

In the previous chapter we had already obtained the Dirac equation in geometric algebra as:

$$e^\beta \nabla_\beta \psi^\dagger = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\,\psi^\dagger$$

$$(\nabla_\alpha \psi)e^\alpha = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\,\psi$$

One applying to the reverse of the wavefunction [73] and the other to the wavefunction but giving both the same results.

5. The Energy-momentum relation create a negative energy whose value is exactly the one expected for the Dark Energy

In Annexa A3 and A3.1 we make a study regrading dark energy. I will make here a summary. Please, check it for details.

The most identifiable effect of the dark energy is the existence of the cosmological constant Λ [84]. Which value [84] is in the order of:

$$\Lambda = 1.1056E - 52m^{-2}$$

It is to be remarked that although being an effect of energy, the dimensions are Length^{-2}.

If we use the energy-momentum relation coming from the present paper (from chapter 2 on):

$$E^2 = m^2 c^4 + p^2 c^2 - R\hbar^2 c^2$$

You can check that there is a new term that reduces the energy of each particle depending on the Ricci scalar (the space-time curvature in the place where the particle is lying). This term has the units of energy squared as it has to be coherent with the rest of the equation.

If we want this term to have units of Length^{-2}, we have to make the following escalation. I call it escalation, because we are not modifying at all the equation. We are just dividing by constants, not adding new variables. The equation is the same. In fact, as if some authors do, I used all the constants equal to 1, the following operation would not even be necessary:

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^4}{\hbar^2 c^2} + \frac{p^2 c^2}{\hbar^2 c^2} - \frac{R\hbar^2 c^2}{\hbar^2 c^2}$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

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The equation above has the same meaning as the original but in another units. Here the energy squared has the units Length^{-2}. So, the last term (-R) represents the square of the reduction of the energy (a negative energy) that apply to every particle in the units Length^{-2}.

What we do in Annexa A3 and A3.1 is to calculate the value of this -R considering different possible metrics of the universe (Interior Schwarzschild metric and exterior Schwarzschild metric via the Kretschmann scalar. As commented this R represents a reduction of energy (a negative energy) in the units Length^{-2}.

We get to the following results:

$$R_{intSchw} = 1.603E - 52m^{-2}$$
$$R_{intSchw1/3} = 0.5345E - 52m^{-2}$$
$$R_{extSchw} \rightarrow \sqrt{K} = 1.852E - 52m^{-2}$$

While the cosmological constant [84] is in the order of:

$$\Lambda = 1.1056E - 52m^{-2}$$

You can see that the value of R is (even considering different metrics) is in the order of Λ. This means, this equation obtained in chapter 2 from the present paper:

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^4}{\hbar^2 c^2} + \frac{p^2 c^2}{\hbar^2 c^2} - \frac{R\hbar^2 c^2}{\hbar^2 c^2}$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Has led (not added ad-hoc, it has appeared directly in the equations) to an element (-R) that reduces the energy of the particles in the exact same value as expected to be considered a candidate for the Dark Energy.

6. Conclusions

In this paper we have obtained a variation of the energy-momentum relation:

$$E^2 = m^2 c^4 + p^2 c^2$$

With an element more that depends on the Ricci scalar curvature:

$$E^2 = m^2 c^4 + p^2 c^2 - R\hbar^2 c^2$$

This leads to a variation in the definition the energy of a mass at rest:

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But it can have its importance in high gravitational fields as a limit to the curvature is

imposed:

$$R < \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2}$$

Also, we have got the variation Klein-Gordon equation in flat space-time:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} - R \right) \psi(t, \boldsymbol{x}) = 0$$

And in curved space time:

$$\frac{-1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \psi \right) + \left(\frac{m^2 c^2}{\hbar^2} - R \right) \psi = 0$$

The same way, we have obtained a variation to the Dirac equation, both in flat and curved space time:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R\psi}$$

$$i\gamma^\mu D_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R\psi}$$

The last point studied is how the obtained equation:

$$E^2 = m^2 c^4 + p^2 c^2 - R\hbar^2 c^2$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Has an element (R, representing the Ricci scalar curvature) that here is acting reducing the energy of the particles. It represents a reduction of the Energy but in units Length^{-2} in the latter equation. This value of R is calculated (considering different metrics for the universe -interior and exterior Schwarzschild metric-) to find that it corresponds almost exactly with the expected value of the cosmological constant (the effects created by the Dark Energy):

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While the cosmological constant is in the order of:

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[76] https://en.wikipedia.org/wiki/Generation_(particle_physics)

[77] https://arxiv.org/abs/1910.11085

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[83] https://link.aps.org/accepted/10.1103/PhysRevD.99.084021

[84] https://en.wikipedia.org/wiki/Cosmological_constant

[85] https://en.wikipedia.org/wiki/Interior_Schwarzschild_metric

[86] https://en.wikipedia.org/wiki/Kretschmann_scalar

A1. Annex A1. Comments regarding single and squared wavefunctions

In chapter 3 we have taken this equation from literature:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2}\right)\psi(t, \boldsymbol{x}) = 0$$

You can see that is a squared equation applying to a single wavefunction. In standard algebra you cannot apply it to the wavefunction squared as the wavefunction squared is normally a probability (a number) or maximum a vector related to probability current. In geometric algebra you can do it like that because the product of a multivector of 8 components (this is the wavefunction by its reverse will give another multivector). So, you can work with the original wavefunction of with its squared with no issues and being completely coherent.

This is the reason in geometric algebra, the Klein-Gordon equation can be applied to the square of the wavefunction with no issues (∇_α applies to the left in this case).

$$e^\beta \nabla_\beta \left((\psi^\dagger \psi)\nabla_\alpha e^\alpha\right) = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\psi^\dagger \psi \sqrt{\frac{m^2 c^2}{\hbar^2} - R}$$

And as commented, it is almost immediate to obtain Dirac equation:

$$e^\beta \nabla_\beta \psi^\dagger = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\psi^\dagger$$

$$(\nabla_\alpha \psi)e^\alpha = \sqrt{\frac{m^2 c^2}{\hbar^2} - R}\psi$$

It is like there are two different worlds regarding the reality. One is the square of the wavefunction (the collapse of the wavefunction), that represents the events that the observers agree that have happened. What we normally call reality or macroscopic reality.

And the other one is the square root of the reality (or the single wavefunction not squared) where what we normally call "Quantum mechanics effects" apply. And it is a world where everything can happen, but it cannot be observed until, this world (or wavefunction) is squared creating what we call "real events" that can be confirmed by observers to have occurred in the "squared world" or reality.

Probably the issue that all elements, instruments… we use (including ourselves) belong to the squared world makes us call it reality, because is the world where we can interact, observe and modify. The square root of the reality (the original wavefunction not squared) is something that we consider have "strange" effects just because of the fact that it is in a world that we cannot control.

And for me it is important that anytime we apply an equation to the wavefunction, we know if it has to be applied to the squared wavefunction (as it should be in the Klein-Gordon equation) or to the non-squared wavefunction (as in the Dirac equation).

Coming back to this, in most of my papers I have considered the time to be the trivector in geometric algebra. Whereas our APS [43] friends consider it to be the scalars. I guess both are right. In the non-squared world probably the trivector and (also the scalars?) could represent what we call the dimension of time. In the squared world (the reality), the trivector has been squared to a negative scalar (and probably from that moment on the time could be represented by negative scalars as APS [43] friends say). But if due to asymmetries when you square a wavefunction, you still have a coefficient for the trivector different from zero, it could be still representing the time.

A2. Annex A2. The electromagnetic trivector

I do not want to miss this opportunity to talk about the electromagnetic trivector. This trivector affects the particles but not changing its "average" speed or direction (as the electric and magnetic moments do).

It has two effects:

- One is changing the orientation of the particle (affecting the bivector planes that represent its orientation).
- It also affects the direction and speed but in an "oscillatory" way. This means it can create momentary changes in the direction and speed in that go in the opposite direction moments later. So it does not affect the "average" trajectory or speed.

This makes the trivector almost invisible, as the effects it has, or either they cannot be measured directly (as orientation of the particle) or are considered noise or uncertainties (for the oscillatory movement). In fact, ipse David Hestenes [77] considers that the zitterbewegung (oscillatory movement) of the electron is related to geometric algebra considerations (being the trivector one of the mostly iconic elements of geometric algebra by definition).

But the issue is that the presence of an electromagnetic trivector could explain effects as the quantum entanglement or certain uncertainties in quantum mechanics.

Regarding the electromagnetic trivector you can find more info in [6][26][47].

A3. Annex A3. Considerations about the Dark Energy

I will include here the same study performed in the paper [73] in its annexa A7 and A7.1.

The most identifiable effect of the dark energy is the existence of the cosmological constant Λ [84]. Which value [84] is in the order of:

$$\Lambda = 1.1056E - 52m^{-2}$$

It is to be remarked that although being an effect of energy, the dimensions are Length[-2].

In the present paper, we have obtained the following energy-momentum relation coming from the chapters 7 to 9 of the paper [73]:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

You can check that there is a new term that reduces the energy of each particle depending on the Ricci scalar (the space-time curvature in the place where the particle is lying). This term has the units of energy squared as it has to be coherent with the rest of the equation.

If we want this term to have units of Length[-2], we have to make the following escalation. I call it escalation, because we are not modifying at all the equation. We are just dividing by constants, not adding new variables. The equation is the same. In fact, as if some authors do, I used all the constants equal to 1, the following operation would not even be necessary:

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^4}{\hbar^2c^2} + \frac{p^2c^2}{\hbar^2c^2} - \frac{R\hbar^2c^2}{\hbar^2c^2}$$

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

The equation above has the same meaning as the original but in another units. Here the energy squared has the units Length[-2]. So, the last term (-R) represents the square of the reduction of the energy (a negative energy) that apply to every particle in the units Length[-2].

So, let's calculate this value and check if it corresponds to the value of the cosmological constant Λ. This means, let's check if this term can be the origin of the "dark energy".

To calculate the Ricci scalar, we have to decide which metric to apply to the universe. The most appropriate for an isotropic universe would be the interior Schwarzschild metric [85].

According to some authors the Ricci scalar of the interior Schwarzschild metric is:

$$R = \frac{1}{3}\frac{(8\pi G)}{c^2}(\rho + 3p + \cdots)$$

Being ρ the Energy density (in kg/m³) and p the pressure. The three points represent some other elements that normally are considered small compared to the previous ones.

In other references [83] -probably depending on different assumptions or definitions-, the Ricci scalar of the interior Schwarzschild metric does not include the 1/3 factor [83].

$$R = \frac{8\pi G}{c^2}(\rho + 3p + \cdots)$$

As it is just a factor, and what we want is to calculate an order of magnitude, let's use the latter equation that does not include it. Anyhow, as the relation between both equations is pretty straight forward, it can be added if it is discovered that the one with 1/3 is the correct one.

For simplification, let's consider that the pressure is zero -or very low compared to energy density- and the points elements can be neglectable. This leads to:

$$R = \frac{8\pi G\rho}{c^2}$$

So, to solve the equation, we need to know the value of the energy density ρ.

The value of ρ (considering that corresponds to the critical density) is calculated using the Hubble constant [81][82][84]. Its value is:

$$H = \frac{67.66\frac{km}{s}}{1Mpc} = \frac{67.66\frac{km}{s}}{3.086E19km} = 2.192E - 18s^{-1}$$

Using the equation [82], we can calculate the density as:

$$\rho = \frac{3H^2}{8\pi G} = \frac{3(2.192E - 18s^{-1})^2}{8\pi(6.67E - 11m^3kg^{-1}s^{-2})} = 8.598E - 27\frac{kg}{m^3}$$

And now, we can introduce it in:

$$R = \frac{8\pi G\rho}{c^2}$$

$$R = \frac{8\pi(6.67E - 11m^3kg^{-1}s^{-2})\left(8.598E - 27\frac{kg}{m^3}\right)}{\left(299792458\frac{m}{s}\right)^2} = 1.603E - 52m^{-2}$$

That you can see that is very near (or at least of the same magnitude) as the cosmological constant Λ:

$$\Lambda = 1.1056E - 52m^{-2}$$

So, this R representing an energy to be subtracted to the particles according to:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Is a perfect candidate for this dark energy.

$$R = 1.603E - 52m^{-2}$$
$$\Lambda = 1.1056E - 52m^{-2}$$

Not much more to say here, the numbers speak by themselves. The energy-momentum relation equation obtained in the present paper, has led to an appearance of a reduction of energy of the particles (a negative energy) that is a perfect candidate to correspond to the Dark Energy.

A3.1 Annex A3.1 Other calculations for R that lead to similar results

In A.3 we have calculated:

$$R = 1.603E - 52m^{-2}$$

Using the equation of R that does not include the 1/3 factor. Just for info, if we had used the equation for R that includes the 1/3 factor, we would have gotten:

$$R = \frac{1}{3}(1.603E - 52m^{-2}) = 0.5345E - 52m^{-2}$$
$$\Lambda = 1.1056E - 52m^{-2}$$

That is also in the order of Λ although not so precise. But the idea that the concept that this negative R in the following equation could be the candidate for the dark energy keeps being the same:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Another thing we could do is to use another metric, to see if we obtain a similar result or not. Instead of using the interior Schwarzschild metric, let's use the exterior Schwarzschild metric.

In the exterior Schwarzschild metric, the value of the Ricci scalar is zero. But we can use the Kretschmann scalar instead [86].

$$K = \frac{48G^2M^2}{c^4r^6}$$

The Kretschmann scalar is a quadratic element. Its units are Length^{-4}. The Ricci scalar is a square element (Length^{-2}). So, to convert the Kretschmann scalar to an equivalent Ricci scalar, we have to take the square root:

$$R \to \sqrt{K} = \frac{\sqrt{48}GM}{c^2r^3}$$

Now, considering spherical symmetry (as in fact it is the Schwarzschild metric) we can make some operations to obtain the equation as dependent of ρ (the energy density in kg/m^3).

$$R = \frac{\sqrt[2]{48}G}{c^2}\frac{M}{r^3} = \frac{\sqrt[2]{48}G}{c^2}\frac{M}{r^3}\frac{\left(\frac{4}{3}\pi\right)}{\left(\frac{4}{3}\pi\right)} = \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\frac{M}{\frac{4}{3}\pi r^3} = \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\frac{M}{V}$$
$$= \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\rho$$

So,

$$R = \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\rho$$

Now, we apply the value of ρ we have calculated in Annex A3:

$$R = \frac{\sqrt[2]{48}G}{c^2}\frac{M}{r^3} = \frac{\sqrt[2]{48}G}{c^2}\frac{M}{r^3}\frac{\left(\frac{4}{3}\pi\right)}{\left(\frac{4}{3}\pi\right)} = \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\frac{M}{\frac{4}{3}\pi r^3} = \frac{\sqrt[2]{48}\left(\frac{4}{3}\pi\right)G}{c^2}\rho$$
$$= \frac{29.020(6.672E - 11\frac{m^3}{s^2kg})}{\left(299792458\frac{m}{s}\right)^2}\ 8.598E - 27\frac{kg}{m^3} = 1.852E - 52m^{-2}$$

So, we have:

$$R = 1.852E - 52m^{-2}$$
$$\Lambda = 1.1056E - 52m^{-2}$$

That again is in the order of Λ, even having an assumption of a different metric.

In fact, the Ricci scalar in the exterior and the interior Schwarzschild metric are in fact very similar (see Annex A3):

$$R_{extSchw} = 1.852E - 52m^{-2}$$
$$R_{intSchw} = 1.603E - 52m^{-2}$$
$$R_{intSchw1/3} = 0.5345E - 52m^{-2}$$
$$\Lambda = 1.1056E - 52m^{-2}$$

One point to be commented also is the possibility that these equations are "circular". In Annex A3 it is calculated the critical density. This density is defined as the density needed to have a near flat space and is calculated using the scape velocity equation. It could be the Ricci scalar curvature was another representation of this "escape velocity" so we are incurring in some kind of circular reference. This is to be studied.

One last comment is regarding the pressure in the equation of the Ricci scalar for the interior Schwarzschild metric:

$$R = \frac{8\pi G}{c^2}(\rho + 3p + \cdots)$$

A study should be done of how it affects when it cannot be considered zero and this could be related to the non-exact equality (although yes in the order of magnitude) between Λ and R in the previous calculations.

Also, we could consider the not at rest particle case with a linear moment different from zero in the following equation:

$$E^2 = m^2c^4 + p^2c^2 - R\hbar^2c^2$$

$$\frac{E^2}{\hbar^2c^2} = \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

But the negative element -R will be the same whether the linear momentum (p) is zero or not in this equation. But it could affect indirectly if the linear momenta of the different particles are affecting the pressure value in the equation of R (via the square of momenta) and so really affecting the final value of R.