The infinite series \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} \) on the Lviv Scottish book is bounded

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“In 2017, I managed to solve a problem from the “Lviv Scottish book. The problem had a prize of “butelka miodu pitnego” (a bottle of honey mead). Today, while I was in Warsaw, some representatives from Lviv, Ukraine came (by train, as the Ukraine airspace is obviously closed) I was very touched and honored to unexpectedly receive the prize in person.”

Terence Tao

Abstract

In this article we prove that the infinite series \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} \) on the Lviv Scottish book is bounded, consequently it is convergent.

Notation and reminder

\( \mathbb{N}^* := \{1,2,3,4,\ldots\} \) the natural numbers.

\( \mathbb{Z} := \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\} \) the integers.

\( \mathbb{R} \) : the set of real numbers and \( \mathbb{R} \setminus \mathbb{Q} \) : the set of irrational numbers.

\( ]0,1[ := \{0 < x < 1 : x \in \mathbb{R}\} \) the open interval with endpoints 0 and 1.

\( |x| := \max\{-x,x : x \in \mathbb{R}\} \) the absolute value of \( x \).

\( \forall \) : the universal quantifier and \( \exists \) : the existential quantifier.

For more details about the infinite series, we refer the reader and our students to [4] and to [5].
Introduction

Is the infinite series \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} \) is convergent? The problem was posed on 22.06.2017 by PhD students of H.Steinhaus Center of Wroclaw Polytechnica. The promised prize for solution is a bottle of drinking honey, see [1] of the Lviv Scottish book. This problem was solved by Terence Tao on 29.09.2017 [2] who is honored on 09.08.2023 [3]. In this article we show that this infinite series is bounded, consequently it is convergent.

**Lemma.** \( \forall \, n \in \mathbb{N}^* \) we have \( 0 < |\sin(n)| < 1 \).

**Proof.** \( \forall \, n \in \mathbb{N}^* \) we have \( 0 \leq |\sin(n)| \leq 1 \), and \( n \not\in \{\frac{k\pi}{2} : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \), thus \( 0 < |\sin(n)| < 1 \).

**Main Theorem.** The infinite series \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} \) is bounded.

**Proof.** Indeed, \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} = \sum_{n=1}^{+\infty} |\sin(n)|^{n/\sqrt{n}} \), and \( \forall \, n \in \mathbb{N}^* \) we have \( 0 < |\sin(n)| < 1 \) and \( n/\sqrt{n} \geq 1 \), this implies that \( 0 < \frac{|\sin(n)|}{\sqrt{n}} < 1 \), then \( \exists \, \alpha, \beta \in ]0,1[ \) such that \( \alpha = \min\{|\sin(n)|^{n/\sqrt{n}} : n \in \mathbb{N}^*\} \) and \( \beta = \max\{|\sin(n)|^{n/\sqrt{n}} : n \in \mathbb{N}^*\} \), then \( \sum_{n=1}^{+\infty} \alpha^n < \sum_{n=1}^{+\infty} |\sin(n)|^{n/\sqrt{n}} < \sum_{n=1}^{+\infty} \beta^n \), thus \( \alpha \frac{1-\alpha}{1-\beta} < \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} < \frac{\beta}{1-\beta} \). Consequently we have \( \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} < +\infty \).

**References**


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