Abstract We have constructed a pitch structure. In this paper, we define a binary relation on the set of steps, thus the set become a circle set. And we define the norm of a key transpose. To apply the norm, we define a scale function on the circle set. Hence we may construct the 2-pitch structure over the circle set.

1. Introduction

In [2], we have constructed a pitch structure $\mathbf{M}$ over a circle set $P$. A pitch structure is a partial structure, and its underlying set is a finite set, see definition 2.11 for more details. A circle set is finite set equipped with the binary relation ‘$\ast$’, cf. definitions 2.5 and 2.6 and proposition 3.1.

Let $P := \{p_0 \ast p_1 \ast \cdots \ast p_{n-1} \ast p_0\}$ be a circle set, and $\{\mathbf{M}_i\}$ a set of pitch structures over $P$ such that $\tau_{\mathbf{M}_i} = p_i$ for every $p_i$. Then we have that $\{\mathbf{M}_i\}$ and $\{SS(\mathbf{M}_i)\}$ are circle sets, see definitions 2.11 and 2.12 and proposition 3.1 for more details.

And we define a scale function $\lambda$ in definition 3.3. Then we have that $\mathbf{M}_i \equiv \mathbf{M}_j$ implies $\lambda(SS(\mathbf{M}_i), SS(\mathbf{M}_j)) = \cdot$ for $\mathbf{M}_i \ast \mathbf{M}_j \in \{\mathbf{M}_i\}$, see proposition 3.2. If we assume that $\kappa_i : SS(\mathbf{M}_i) \rightsquigarrow SS(\mathbf{M}_j)$ is a regular key transpose for $\mathbf{M}_i \ast \mathbf{M}_j \in \{\mathbf{M}_i\}$, then we have that $\lambda(SS(\mathbf{M}_i), SS(\mathbf{M}_j)) = \cdot$, see proposition 3.3 for more details.

We may construct a pitch structure $\hat{\mathbf{M}} := \{(SS(\mathbf{M}_i)) \cup \mathbf{S}, \lambda, \tau, \mathbf{S}, \ast\}$ over the circle set $\{SS(\mathbf{M}_i)\}$, see proposition 3.4 for more details. We call $\mathbf{M}$ 2-pitch structure. And we may obtain an $n$-pitch structure in this way.

2. Preliminaries

2.1. Universal Algebra. We recall some definitions in universal algebra.

Definition 2.1 ([1, 3]). An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) language provided that
\begin{itemize}
  \item \(L\) is a nonempty set,
  \item \(\sigma : L \rightarrow \mathbb{Z}\) is a mapping.
\end{itemize}

A language \((L, \sigma)\) is denoted by \(\mathcal{L}\). If \(f \in \mathcal{L}\) and \(\sigma(f) \geq 0\) then \(f\) is called an \textit{operation symbol}, and \(\sigma(f)\) is called the \textit{arity} of \(f\). If \(r \in \mathcal{L}\) and \(\sigma(r) < 0\), then \(r\) is called a \textit{relation symbol}, and \(-\sigma(r)\) is called the \textit{arity} of \(r\). A language is said to be \textit{algebraic} if it has no relation symbols.

**Definition 2.2** ([1]). Let \(X\) be a nonempty class and \(n\) a nonnegative integer. Then an \(n\)-ary \textit{partial operation} on \(X\) is a mapping from a subclass of \(X^n\) to \(X\). If the domain of the mapping is \(X^n\), then it is called an \(n\)-ary \textit{operation}. And an \(n\)-ary \textit{relation} is a subclass of \(X^n\) where \(n > 0\). An operation (relation) is said to be \textit{unary}, \textit{binary} or \textit{ternary} if the arity of the operation (relation) is 1, 2 or 3, respectively. And an operation is called \textit{nullary} if the arity is 0.

**Definition 2.3** ([1]). An ordered pair \(\mathcal{A} := \langle A, \mathcal{L} \rangle\) is said to be a \textit{structure} of a language \(\mathcal{L}\) if \(A\) is a nonempty class and there exists a mapping which assigns to every \(n\)-ary operation symbol \(f \in \mathcal{L}\) an \(n\)-ary operation \(f^A\) on \(A\) and assigns to every \(n\)-ary relation symbol \(r \in \mathcal{L}\) an \(n\)-ary relation \(r^A\) on \(A\). If all operation on \(A\) are partial operations, then \(A\) is called a \textit{partial structure}. A (partial) structure \(\mathcal{A}\) is said to be a \textit{(partial)algebra} if the language \(\mathcal{L}\) is algebraic.

**Definition 2.4** ([1, 3]). Let \(A, B\) be (partial) structures of a language \(\mathcal{L}\). A mapping \(\varphi : A \rightarrow B\) is said to be a \textit{homomorphism} provided that
\[
\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n)) \quad \text{for every } n\text{-ary operation } f;
\]
\[
\varphi(r^A(a_1, \ldots, a_n)) \subseteq r^B(\varphi(a_1), \ldots, \varphi(a_n)) \quad \text{for every } n\text{-ary relation } r.
\]

A homomorphism \(\varphi\) is called an \textit{isomorphism} if \(\varphi\) is bijective.

2.2. Pitch Structures. We recall some important definitions, see paper [2] for more details.

**Definition 2.5** ([2]). Suppose that \(P\) is a nonempty finite set. We may define a binary relation ‘\(\ast\)’ on \(P\) as follows. For every \(s \in P\),
\begin{itemize}
  \item there is exactly one \(u \in P\) such that \(u \ast s\), and
  \item there is exactly one \(v \in P\) such that \(s \ast v\).
\end{itemize}

**Definition 2.6** ([2]). A \textit{circle set} is a nonempty finite set equipped with the binary relation ‘\(\ast\)’ defined in \textit{Definition 2.5}. Let \(P\) be a circle set. Then a bijective mapping \(\delta : P \rightarrow P\) is said to be a \textit{shift} if \(\delta\) preserves the order of \(P\), i.e., \(\delta(p_1) \ast \delta(p_2)\) if and only if \(p_1 \ast p_2\).

**Definition 2.7** ([2]). Suppose that \(P\) is a circle set. Let \(\tau := p\) for an arbitrary \(p \in P\). We call \(\tau\) a \textit{tonic} of \(P\).

**Definition 2.8** ([2]). Suppose that \(P\) is a circle set. Let \(\mathcal{S}\) be the set \(\{\ast, \circ, \otimes\}\). We may define a function \(\lambda : P \times P \rightarrow \mathcal{S}\) given by
\[
\lambda(p, p') = \begin{cases} 
\ast & \text{if } p \ast p', \\
\circ & \text{otherwise.}
\end{cases}
\]

And the elements of the set \(\mathcal{S}\) is called \textit{scales}. 


**Definition 2.9** ([2]). Suppose that $P$ is a circle set. Let $\sharp$ be a unary relation on $P$ such that

1. $\lambda(\sharp(s), \sharp(p)) = \lambda(s, p)$;

2. $\lambda(s, \sharp(p)) = \begin{cases} \land & \text{if } \lambda(s, p) = \land, \\ \lor & \text{if } \lambda(s, p) = \lor \end{cases}$

3. $\lambda(\sharp(s), p) = \begin{cases} \land & \text{if } \lambda(s, p) = \land, \\ \lor & \text{if } \lambda(s, p) = \lor \end{cases}$

for every $s, p \in P$ with $s \neq p$.

**Definition 2.10** ([2]). Suppose that $P$ is a circle set. Let $\flat$ be a unary relation on $P$ such that

1. $\lambda(\flat(s), \flat(p)) = \lambda(s, p)$;

2. $\lambda(s, \flat(p)) = \begin{cases} \land & \text{if } \lambda(s, p) = \land, \\ \lor & \text{if } \lambda(s, p) = \lor \end{cases}$

3. $\lambda(\flat(s), p) = \begin{cases} \land & \text{if } \lambda(s, p) = \land, \\ \lor & \text{if } \lambda(s, p) = \lor \end{cases}$

for every $s, p \in P$ with $s \neq p$.

**Remark 2.1.** In fact, that $\sharp$ and $\flat$ are not real unary relations.

**Definition 2.11** ([2]). A partial structure $M := \langle M, \mathcal{L} \rangle$ of the language $\mathcal{L}$ is called a **pitch structure** over a circle set $P$ provided that the underlying set $M = P \cup S$ where $P$ equipped with $\mathcal{L}$ is a circle set [definition 2.6], and the language is defined to be the set $\mathcal{L} := \{ \lambda, \tau, S, \emptyset \}$ where $\lambda$ is a partial binary operation defined in definition 2.8, $\tau$ is a binary relation defined in definition 2.5, $\emptyset$ is a nullary operation defined in definition 2.7, and $S = \{ \land, \lor, \land \}$ is the set of nullary operations defined in definition 2.8.

**Definition 2.12** ([2]). Let $M$ be a pitch structure over a circle set $P$, $|P| = n$, and $\tau := m_0$ for $m_0 \in P$. Then we define $SS_{\tau_\mathcal{L}}(M)$ to be the following sequence

1. $\{ \lambda(m_0, m_1), \lambda(m_1, m_2), \ldots, \lambda(m_{n-2}, m_{n-1}), \lambda(m_{n-1}, m_0) \}$,

if we have $m_0 \land m_1 \land m_2 \land \cdots \land m_n \land m_2 \land m_0 \in P$. And the sequence $SS_{\tau_\mathcal{L}}(M)$ is called a **step** of the pitch structure $M$ at the tonic $m_0$.

**Definition 2.13** ([2]). Suppose that $M$ is a pitch structure over a circle set $P$, and the tonic $\tau = m_0$. Then the ordered pair $\langle \tau_M, SS_{\tau_\mathcal{L}}(M) \rangle$ is called the **key** of $M$.

**Definition 2.14** ([2]). Suppose that $M, N$ are pitch structures over a circle set $P$, and $\tau_M = m_i$, $\tau_N = m_j$ for $m_i, m_j \in P$. Let $\delta$ be a shift [definition 2.6] which assigns $m_j$ to $m_i$. Then a bijective mapping $\kappa: SS_{\tau_M}(M) \mapsto SS_{\tau_N}(N)$ is called a **key transpose** along $\delta$ provided that $\kappa$ assigns $\lambda_N(\delta(m), \delta(m'))$ to $\lambda_M(m, m')$ for every $m, m' \in P$ with $m \land m'$. 
3. A 2-Pitch Structure

**Definition 3.1.** Let \( v : S \times S \to \{-1, 0, 1\} \) be a function defined as follows:

\[
(3.1) \quad v(x, y) = \begin{cases} 
1 & \text{if } (x, y) = (-, -) \\
-1 & \text{if } (x, y) = (, -) \\
0 & \text{otherwise},
\end{cases}
\]

where the set \( S \) is defined in definition 2.8.

**Definition 3.2.** Suppose that \( M, N \) are two pitch structures over a circle set \( P \), and \( P := \{ p_0 \ominus p_1 \ominus \cdots \ominus p_{n-1} \ominus p_0 \} \). Let \( \kappa : SS(M) \mapsto SS(N) \) be a key transpose along a shift \( \delta \). Then the integer

\[
(3.2) \quad \| \kappa \| = \sum_{i=0}^{n-1} v(\lambda_M(p_i, p_{i+1} \mod n), \lambda_N(\delta(p_i), \delta(p_{i+1} \mod n)))
\]

is called norm of \( \kappa \), where \( \lambda \) is defined in definition 2.8, \( v \) is defined in definition 3.1, \( \delta \) is defined in definition 2.6, \( SS(M) \) is defined in definition 2.12, and \( \kappa \), that is defined in definition 2.14, assigns \( \lambda_N(\delta(p_i), \delta(p_{i+1} \mod n)) \) to \( \lambda_M(p_i, p_{i+1} \mod n) \).

**Proposition 3.1.** Let \( P := \{ p_0 \ominus p_1 \ominus \cdots \ominus p_{n-1} \ominus p_0 \} \) be a circle set. Suppose that \( \{ M_i \}_{0 \leq i < n-1} \) is a set of pitch structures over the circle set \( P \), and \( \tau M_i = p_i \) for every \( i \in \{ 0, \ldots, n-1 \} \) where \( \tau \) is defined in definition 2.11. Then the sets \( \{ M_i \}_{0 \leq i < n-1} \) and \( \{ SS(M_i) \}_{0 \leq i < n-1} \) are circle sets.

**Proof.** We define \( M_i \ominus M_j, SS(M_i) \ominus SS(M_j) \) if \( p_i \ominus p_j \in P \). Then this is an immediate consequence of definitions 2.5 and 2.6. \( \square \)

**Definition 3.3.** Let the notations be as in proposition 3.1, and \( \overline{P} := \{ SS(M_i) \}_{0 \leq i < n-1} \). Suppose that \( SS(M_i) \ominus SS(M_j) \), and \( \kappa_i : SS(M_i) \mapsto SS(M_j) \) is a key transpose along the shift \( \delta_i \), which assigns \( \tau M_i \) to \( \tau M_j \) for all \( i, j \in \{ 0, \ldots, n-1 \} \) with \( j = (i + 1) \mod n \). Then let \( \overline{\lambda} : \overline{P} \times \overline{P} \to S \) be a function given by

\[
(3.3) \quad \overline{\lambda}(SS(M_i), SS(M_j)) = \begin{cases} 
- & \text{if } \| \kappa_i \| \geq 0 \\
0 & \text{if } \| \kappa_i \| < 0,
\end{cases}
\]

where \( \| \kappa_i \| \) is as defined in definition 3.2.

**Proposition 3.2.** Let the notations be as in definition 3.3. We have that \( M_i \equiv M_j \) implies \( \overline{\lambda}(SS(M_i), SS(M_j)) = - \) for \( M_i \ominus M_j \in \{ M_i \}_{0 \leq i < n-1} \).

**Proof.** By [2, proposition 3.1], we have that \( SS(M_i) = SS(M_j) \). It follows that \( \| \kappa_i \| = 0 \) by equations (3.1) and (3.2). Therefore, we have that \( \overline{\lambda}(SS(M_i), SS(M_j)) = - \) by equation (3.3). \( \square \)

**Proposition 3.3.** Let the notations be as in definition 3.3. If \( \kappa_i \) is regular [2, definition 3.12], then \( \overline{\lambda}(SS(M_i), SS(M_j)) = - \) for \( SS(M_i) \ominus SS(M_j) \in \overline{P} \).

**Proof.** Since [2, lemmas 3.1 and 3.2], we have that if \( \kappa_i \) is regular, then

- there exist a \( \lambda \)-shrink if and only if there exists a \( \lambda \)-stretch, and
- there exist a \( \lambda \)-shrink if and only if there exists a \( \lambda \)-stretch.

Hence there exists \( \kappa_i : - \mapsto - \) if and only if there exists \( \kappa_i : - \mapsto - \). This implies \( \| \kappa_i \| = 0 \) by equation (3.2). This completes the proof by equation (3.3). \( \square \)
Remark. It is obvious that the converses of propositions 3.2 and 3.3 do not hold.

Proposition 3.4. Let the notations be as in definition 3.3. And let $\tau = SS(M_i)$ for an $i \in \{0, \ldots, n-1\}$. Then the partial structure $\mathbf{M} := (\mathbf{P} \cup S, \lambda, \tau, S, \phi)$ is a pitch structure over the circle set $\mathbf{P}$. And the pitch structure $\mathbf{M}$ is called 2-pitch structure.

Proof. The proposition follows from definitions 2.11 and 3.3 and proposition 3.1. \hfill \Box

References


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