Zeta Function

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Abstract
This article delves into the properties of the Riemann zeta function, providing a demonstration of the existence of a sequence of zeros $z_k$, where $\lim \text{Re}(z_k) = 1$.

The exploration of these mathematical phenomena contributes to our understanding of complex analysis and the behavior of the zeta function on the critical line.

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MSC Classification: 11-11, 11Mxx

1 Introduction

In this article, I present a demonstration revealing that the Riemann zeta function possesses a sequence of zeros $z_k$ with the property $\lim \text{Re}(z_k) = 1$.

This is established by assuming the convergence of the series in the region $\text{Re}(s) > \rho$:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} < +\infty$$

(1.1)

This assumption is equivalent to the absence of zeros of the Riemann zeta function $(\zeta(s))$ in the region where $\text{Re}(s) > \rho$, a fact proven in [1].

Under the condition, I prove that the following implication holds:

$$\frac{\zeta(s)}{\zeta(1-s)} = s \int_{0}^{\infty} \frac{1}{x^{s+1}} \frac{\sin(2\pi x)}{\pi} \, dx$$

(1.2)

leading to a contradiction.
The proof involves the observation that:

\[-\frac{\zeta(s)}{s(s+1)(s+2)}\frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\theta_n(x)}{x^{s+1}} dx\]  

(1.3)

where

\[\phi_n(x) = \int_0^x nu\frac{\mu(n)}{n} du\]  

(1.4)

\[\theta_n(x) = \int_0^x \phi_n(u) du\]  

(1.5)

and

\[\sum_{n=1}^\infty \theta_n(x) = 1 + \frac{\sin(2\pi x)}{2\pi} - x\]  

(1.6)

To establish this result, I utilize the inverse Mellin transform to estimate an upper bound for

\[\sum_{n=M}^\infty \theta_n(x)\]  

(1.7)

This yields:

\[\sum_{n=M}^\infty \theta_n(x) \leq x^{\rho+2} \max\left|\frac{1}{\rho+2+it}\right| \sum_{n=M}^\infty \frac{\mu(n)}{n^{1-\rho-it}}, t \in \mathbb{R} \int_{-\infty}^{\infty} \frac{\zeta(\rho+t)}{(\rho+it)(\rho+1+it)} dt\]  

(1.8)

Consequently, by comparing upper bounds on both sides of the equality, we deduce the contradiction in (1.6). The proof of the inconsistency in (1.2) is straightforward, as it involves a comparison of upper bounds for the functions on either side of the equation, revealing a mismatch.

2 Fundamental Theorems

In this section, I will list some theorems used throughout the article.

**Theorem 2.1.** If \(\varphi(s)\) is analytic in the strip \(a < \text{Re}(s) < b\), and if it tends to zero uniformly as \(\text{Im}(s) \to \pm\infty\) for any real value \(c\) between \(a\) and \(b\), with its integral along such a line converging absolutely, then if

\[f(x) = \mathcal{M}^{-1}\varphi = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \varphi(s) ds,\]

we have that

\[\varphi(s) = \mathcal{M}f = \int_0^\infty x^{s-1} f(x) dx.\]

Conversely, suppose \(f(x)\) is piecewise continuous on the positive real numbers, taking a value halfway between the limit values at any jump discontinuities, and suppose the integral

\[\varphi(s) = \int_0^\infty x^{s-1} f(x) dx\]
is absolutely convergent when $a < \text{Re}(s) < b$. Then $f$ is recoverable via the inverse Mellin transform from its Mellin transform $\varphi$.

Proof. [2]

**Theorem 2.2.** If $\text{Re} > 1$, we have:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

If the zeta function has no zeros in the region $\text{Re}(s) > \rho$, we can extend the equality above to such a region.

Proof. [3]

**Theorem 2.3.** If $0 < \text{Re}(s) < 1$, we have:

$$-\frac{\zeta(s)}{s} = \int_{0}^{\infty} \frac{\{x\}}{x^{s+1}} \, dx$$

Proof. [4]

**Theorem 2.4.** For any natural number $n > 1$, the sum of the values of the Möbius function $\mu(d)$ over all positive divisors of $n$ is given by:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Proof. [1]

3 **Proof**

In the case where $0 < \text{Re}(s) < 1$:

$$\zeta(s) = -s \int_{0}^{\infty} \frac{\{y\}}{y^{s+1}} \, dy \quad (3.1)$$

Thus:

$$-\frac{\zeta(s)}{s} \frac{\mu(n)}{n^{1-s}} = \int_{0}^{\infty} \frac{\{nx\}}{x^{s+1}} \frac{\mu(n)}{n} \, dx \quad (3.2)$$

where $n \in \mathbb{Z}$.

Integrating by parts in equation (3.2), we obtain:

$$-\frac{\zeta(s)}{s(s+1)} \frac{\mu(n)}{n^{1-s}} = \int_{0}^{\infty} \frac{\phi_n(x)}{x^{s+2}} \, dx \quad (3.3)$$
Where:
\[ \phi_n(x) = \int_0^x \{nu\} \frac{\mu(n)}{n} du \]  
(3.4)

Doing one more integration by parts, we have:
\[ -\frac{\zeta(s)}{s(s+1)(s+2)} \frac{\mu(n)}{n^{1-s}} = \int_0^\infty \frac{\theta_n(x)}{x^{s+3}} dx \]  
(3.5)

And
\[ \theta_n(x) = \int_0^x \phi_n(u) du \]  
(3.6)

Using the fact that, for every \( 0 < x < 1 \), we have:
\[ \{x\} = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^\infty \frac{\sin(2\pi nx)}{n} \]  
(3.7)

It follows that:
\[ \sum_{n=1}^\infty \theta_n(x) = \frac{1}{2\pi^2} \left( \frac{\sin(2\pi x)}{2\pi} - x \right) \]  
(3.8)

Indeed, by (3.7) we have:
\[ \phi_n(x) = \frac{x}{2} + \frac{1}{2\pi^2} \sum_{k=1}^\infty \frac{\cos(2\pi nkx) - 1}{k^2} \]  
(3.9)

And
\[ \theta_n(x) = \frac{x^2}{4} + \frac{1}{4\pi^2} \sum_{k=1}^\infty \frac{\sin(2\pi nkx)}{k^3} - \frac{x}{2\pi^2} \sum_{k=1}^\infty \frac{1}{k^2} \]  
(3.10)

Thus:
\[ \sum_{n=1}^\infty \frac{\mu(n)}{n} \theta_n(x) = \frac{x^2}{4} \sum_{n=1}^\infty \frac{\mu(n)}{n} - \frac{x}{2\pi^2} \sum_{n=1}^\infty \frac{\mu(n)}{n^2} \sum_{k=1}^\infty \frac{1}{k^2} + \frac{1}{4\pi^2} \sum_{n,k=1}^\infty \frac{\sin(2\pi nkx)\mu(n)}{n^3k^3} \]  
(3.11)

For:
\[ \sum_{n,k=1}^\infty \frac{\sin(2\pi nkx)\mu(n)}{n^3k^3} = \sum_{l=1}^\infty \frac{\sin(2\pi lx)}{l^3} \sum_{n|l} \mu(n) = \sin(2\pi x) \]  
(3.12)

(The rearrangement of the summations is justified by the uniform convergence of the series)
\[ \sum_{n=1}^\infty \frac{\mu(n)}{n} = 0 \]  
(3.13)

And
\[ \sum_{n=1}^\infty \frac{\mu(n)}{n^3} \sum_{k=1}^\infty \frac{1}{k^3} = 1 \]  
(3.14)
we conclude (3.8).

Using the inverse Mellin transform on (3.5):

$$\theta_{kn}(x) = -\int_{\sigma - i\infty}^{\sigma + i\infty} x^{s+2} \frac{\zeta(s)}{s(s+1)(s+2)n^{1-s}} ds$$  

(3.15)

where $\sigma = \text{Re}(s)$ and $0 < \sigma < 1$. With this:

$$\sum_{n=M}^{M+P} \theta_{kn}(x) = -\int_{\sigma - i\infty}^{\sigma + i\infty} x^{s+1} \frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} ds$$  

(3.16)

Assume that:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1-s}} < +\infty$$  

(3.17)

for $\sigma = \text{Re}s \leq \rho$, where it is known that $\rho < \frac{1}{2} - \epsilon, \epsilon > 0$.

In this case, we have:

$$\sum_{n=M}^{\infty} \theta_{kn}(x) \leq x^{\gamma+2} \max \{ \| \frac{1}{(\gamma + 3 + it)} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^{1-\gamma-it}} \|, t \in \mathbb{R} \} \int_{-\infty}^{\infty} \| \frac{\zeta(\gamma + t)}{(\gamma + it)(\gamma + 1 + it)} \| dt$$  

(3.18)

with $0 < \gamma < \rho$.

Where, by hypothesis:

$$\psi_{\gamma}(M) = \max \{ \| \frac{1}{(\gamma + 3 + it)} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^{1-\gamma-it}} \|, t \in \mathbb{R} \}$$  

(3.19)

$$\lim_{M \to \infty} \psi_{\gamma}(M) = 0$$  

(3.20)

Since:

$$\sum_{n=k}^{\infty} \frac{\mu(n)}{n^{s}} = \frac{M(k)}{k^{s}} - s \int_{k}^{\infty} \frac{M(x)}{x^{s+1}} dx$$

Where:

$$M(x) = \sum_{n=1}^{x} \mu(n)$$

By equation (3.5), we have:

$$-\frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=1}^{M} \frac{\mu(n)}{n^{1-s}} = \int_{0}^{\infty} \frac{1}{x^{s+3}} \sum_{n=1}^{M} \theta_{kn}(x) dx$$  

(3.21)

Note that:

$$\int_{0}^{\infty} \frac{1}{x^{s+3}} \sum_{n=1}^{M} \theta_{kn}(x) dx = \int_{0}^{1} \frac{1}{(x+k)^{s+3}} \sum_{n=1}^{M} \theta_{kn}(x+k) dx + \int_{0}^{1} \frac{1}{x^{s+3}} \sum_{n=1}^{M} \theta_{kn}(x) dx$$  

(3.22)
Using (3.18), we conclude that this difference tends to zero when \( M \to \infty \), if \( \text{Re} \rho < \rho - \epsilon \), for every \( \epsilon > 0 \). Indeed, using (3.18), we conclude that:

\[
\int_0^1 G(x) \sum_{n=M}^{\infty} \theta_n(x) dx + \int_0^1 \frac{1}{x^{\gamma+3}} \sum_{n=M}^{\infty} \theta_n(x) dx < C_\gamma \psi_\gamma (M) \int_0^1 \sum_{k=1}^{\infty} \frac{(x+k)^{\gamma+2}}{(x+k)^{\sigma+3}} dx + C_\rho \psi_\rho (M) \frac{1}{\rho - \sigma}
\]

(3.23)

Where

\[
C_\sigma = \int_{-\infty}^{\infty} \frac{\zeta (\sigma + t)}{\gamma} dt
\]

(3.24)

And the result follows from (3.19).

With this, taking the limit in (3.21), we conclude:

\[
-\frac{\zeta (s)}{\zeta (1-s)} = \int_0^\infty \frac{1}{x^{\gamma+3}} \frac{1}{2\pi} \{ \frac{\sin (2\pi x)}{2\pi} - x \} dx
\]

(3.25)

Where \( 0 < \text{Re} s < \rho \).

However, by analytic continuation, it is concluded that this equality holds for all \( 0 < \text{Re}(s) < 1 \). Performing integrations by parts, we obtain:

\[
\frac{\zeta (s)}{\zeta (1-s)} = s \int_0^\infty \frac{1}{x^{\gamma+1}} \frac{\sin (2\pi x)}{\pi} dx
\]

(3.26)

Defining:

\[
F(s) = \pi \int_0^\infty \frac{\sin (2\pi x)}{x^{\gamma+1}} dx
\]

(3.27)

\( F \) is a holomorphic function in the region \( 0 < \text{Re}(s) < 1 \), and furthermore, \( F(s) = O \left( \frac{1}{s} \right) \), indeed, writing:

\[
F(s) = F_1 (s) + F_2 (s)
\]

(3.28)

Where:

\[
F_1 (s) = \pi \int_0^2 \frac{\sin (2\pi x)}{x^{\gamma+1}} dx
\]

(3.29)

\[
F_2 (s) = \pi \int_2^\infty \frac{\sin (2\pi x)}{x^{\gamma+1}} dx
\]

(3.30)

Note that:

\[
\int_0^2 \cos (2\pi x) x^{-s} dx = \frac{s}{2\pi} \int_0^2 \frac{\sin (2\pi x)}{x^{\gamma+1}} dx
\]

(3.31)

Hence, we conclude that \( F_1 (s) = O \left( \frac{1}{s} \right) \).

Now, observing that \( F_2 (s) = O \left( \frac{1}{s} \right) \),

\[
F_2 (s) = 2^s \pi \int_1^\infty \frac{\sin (\pi x)}{x^{\gamma+1}} dx
\]

(3.32)

\[
\int_1^\infty \frac{\sin (\pi x)}{x^{\gamma+1}} dx = \frac{\pi}{s} \int_1^\infty \frac{\cos (\pi x)}{x^{\gamma}} dx
\]

(3.33)
And
\[
\int_{1}^{\infty} \frac{\cos (\pi x)}{x^s} \, dx = \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{\cos (\pi x)}{x^s} \, dx = \int_{0}^{1} \frac{\cos (\pi x)}{x^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^s} \, dx \quad (3.34)
\]

As the function
\[
\psi(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^s}
\]

is bounded for Re\(s\) > 0 and \(x > -1\), we conclude that \(F_2(s) = O\left(\frac{1}{s}\right)\).

With this result, it can be inferred from equation (3.31) that:

\[
\frac{\zeta(s)}{\zeta(1-s)} = sF(s) = O(1) \quad (3.36)
\]

By the Riemann functional equation:

\[
\frac{\zeta(s)}{\zeta(1-s)} = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = sF(s) = O(1) \quad (3.37)
\]

For every \(s\) in \(0 < \text{Re}(s) < 1\). Absurd, considering:

\[
\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = O(||s||^{\frac{1}{2} - \text{Re}(s)}) \quad (3.38)
\]

Therefore, it is concluded that:

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}
\]

does not converge if Re\(s\) < 1, implying that the zeta function has a sequence of zeros \(\{z_k\}\) such that \(\lim \text{Re}(z_k) = 1\).

4 Conclusion

In this article, I demonstrate that the Riemann zeta function possesses a sequence of zeros, with their real parts converging to 1.

References

