Physically reinterpreted effect in symmetric measurements in quantum optics

Koji Nagata,[1] Do Ngoc Diep,[2] and Tadao Nakamura[3]

[1] *Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 34141, Korea*
E-mail: ko_mi_na@yahoo.co.jp
[2] *Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet road, Cau Giay district, Hanoi, Vietnam*
[3] *Department of Information and Computer Science, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*

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Abstract

We study the mathematical foundations of quantum optics. We consider the case that the uncertainty principle exists because we think both of commutativeness and non-commutativeness and then we derive natural value. On the other hand, we consider the case that the uncertainty principle does not exist because we think only commutativeness and then we derive unnatural value. We propose an experimental accessible inconsistency within quantum optical phenomena in terms of imperfect sources and detectors. In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable in a real experimental situation. Such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers. As a result of our study, a perfect mathematical model for quantum optical phenomena does not exist, at this stage.

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I. INTRODUCTION

Quantum mechanics (cf. [1–7]) gives explanations for the microscopic behaviors of the nature. We see researches concerning the mathematical formulations of quantum mechanics. For example, the mathematical foundations of quantum mechanics are discussed by Mackey [8]. On the quantum logic approach to quantum mechanics is also discussed by Gudder [9]. Conditional probability and the axiomatic structure of quantum mechanics are also reported by Guz [10].

On the other hand, the incompleteness argument to quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [11]. A hidden-variable interpretation of quantum mechanics is a topic of research [2, 3] and the no-hidden-variable theorem is discussed by Bell, Kochen, and Specker [12, 13].

Recently, Nagata, Diep, and Nakamura discuss a novel inconsistency within quantum mechanics without extra assumptions about the reality of observables, using two symmetric measurements. They are free from the order of measurements themselves [14]. As the aim of this paper, we propose an experimental accessible inconsistency within quantum optical phenomena in terms of imperfect sources and detectors based on the argumentations [14]. We hope not only theoretical physicists but also experimental physicists can easily understand our claim.

In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable in a real experimental situation. If we use the quantum predictions by even number $2N$ trials, then the inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

II. EXPLANATIONS FOR THE INCONSISTENCY USING COMMUTING OBSERVABLES

The contradiction in this paper is explained this: If we allow to take both of commutativeness and non-commutativeness in consideration, there is the uncertainty principle, which fact seems to be likely to be quantum theory. And that seems to be natural to have the value of "zero".

On the other hand, we would discuss that the sum rule is equivalent to the product rule for commuting observables. First, we define the functional rule as follows:

$$f(g(O)) = g(f(O)), \tag{1}$$

where O is a Hermitian operator and f, g are appropriate functions. Second, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2), \tag{2}$$

where A_1, A_2 are Hermitian operators. Finally, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{3}$$

This fact above (the sum rule is equivalent to the product rule) is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [3, 14] the following relation between the three rules:

$$\begin{aligned}&\text{The functional rule}\\ &\Leftrightarrow \text{The sum rule}\\ &\Leftrightarrow \text{The product rule}\end{aligned} \tag{4}$$

In fact, the sum rule is equivalent to the product rule for commuting observables.

And then, we can create a novel algebra that might be called commuting observable algebra. The space for commuting observable algebra is different from our common space in which Newton's mechanics is created. In the space holding commuting observable algebra, the operation Addition and the operation Multiplication are the same as each other. In the normal space where Newton's mechanics is held, the operation Addition and the operation Multiplication are different from each other.

The space holding commuting observable algebra in considering only commutativeness is opposite to quantum theory. The reason of the inconsistency is because there is not the uncertainty principle because of not being noncommutative. Besides, holding commuting observable algebra allows unnatural value to be "2". Therefore, we have a contradiction. The point of the contradiction is based on our consideration that we have noncommutative property in quantum operations. This fact is based on the uncertainty principle. Here we must notice theoretically that the space holding commuting observable algebra thinking of only commutativeness is different from the normal space holding Newton's mechanics.

The scenario of this paper tells us that Newton's mechanics is not held when thinking of only commutativeness. The other case is in the space holding commuting observable algebra. On the other hand, in case of holding both of commutativeness and non-commutativeness, the space not permitting commuting observable algebra is allowed, and quantum theory is permitted, and then the uncertainty principle exists.

III. AN OPTICAL EFFECT IN TWO SYMMETRIC AND FINITE PRECISION MEASUREMENTS

Let σ_z be the z-component Pauli observable. It could be defined as follows:

$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of σ_z such that $\sigma_z|\uparrow\rangle = +1|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -1|\downarrow\rangle$. The measured results of trials are either $+1$ or -1 in the ideal case.

When we consider a quantum optical experiment, we have the following relations with the photon polarization states:

$$\begin{aligned}|\uparrow\rangle &\leftrightarrow |H\rangle,\\ |\downarrow\rangle &\leftrightarrow |V\rangle,\end{aligned} \tag{6}$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon and $|V\rangle$ is a quantum state interpreted by a vertically polarized photon.

Let us introduce the random noise admixture $\rho_{\text{noise}}(=\frac{1}{2}I)$ into the quantum states, where I is the two-dimensional identity operator. We consider the noisy quantum states emerged from an imperfect source as follows:

$$\begin{aligned}\rho_1 &= (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}},\\ \rho_2 &= (1-\epsilon)|\downarrow\rangle\langle\downarrow| + \epsilon \times \rho_{\text{noise}}.\end{aligned} \tag{7}$$

The value of $\epsilon(< 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\text{tr}[\rho_1\sigma_z] = +1-\epsilon$ and $\text{tr}[\rho_2\sigma_z] = -1+\epsilon$.

We might be in an inconsistency when the first result is $+1-\epsilon$ by the measured observable σ_z, the second result is $-1+\epsilon$ by the measured observable σ_z, and then we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. This means the uncertainty principle does not exist because we think only commutativeness.

In general, the physical situation is either $[\sigma_z, \sigma_z] \neq 0$ or $[\sigma_z, \sigma_z] = 0$. This means the uncertainty principle exists because we think both of commutativeness and non-commutativeness.

We consider a value V which is the sum of two data in an optical experiment. The measured results of trials are either $+1-\epsilon$ or $-1+\epsilon$. We suppose the number of $-1+\epsilon$ is equal to the number of $+1-\epsilon$. If the number of trials is 2, then we have

$$V = (+1-\epsilon) + (-1+\epsilon) = 0. \tag{8}$$

We derive a general and natural necessary condition of the product $V \times V$ of the value V. In this general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness, and we have

$$V \times V = 0. \tag{9}$$

This is the general and natural necessary condition when we consider both the propositions $[\sigma_z, \sigma_z] \neq 0$ and $[\sigma_z, \sigma_z] = 0$ and we assign simultaneously the different two values ("0" and "1") for the propositions $[\sigma_z, \sigma_z] \neq 0$ and $[\sigma_z, \sigma_z] = 0$. We assign the value "0" for the proposition $[\sigma_z, \sigma_z] \neq 0$.

On the other hand, we can depict the optical experimental data r_1, r_2 as follows: $r_1 = +1-\epsilon$ and $r_2 = -1+\epsilon$. Let us write V as follows:

$$V = r_1 + r_2. \tag{10}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific and unnatural necessary condition under the supposition that the two measured observables are commuting (that is, $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it) and we do not consider the existence of the following proposition $[\sigma_z, \sigma_z] \neq 0$. In this specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the sum rule is equivalent to the product rule [14]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have using commuting observable algebra,

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1 \times r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 \times r_2) \\
&= (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\
&= (+1 - \epsilon)^2 + (-1 + \epsilon)^2 = 2(+1 - \epsilon)^2.
\end{aligned} \tag{11}$$

Thus, we have

$$V \times V = 2(+1 - \epsilon)^2. \tag{12}$$

This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. In this specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (9) and (12) when we consider only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. We derive the inconsistency when we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it.

In summary, we have been in the inconsistency when the first result is $+1-\epsilon$ by measuring the Pauli observable σ_z in the quantum state ρ_1, the second result is $-1+\epsilon$ by measuring the same Pauli observable σ_z in the quantum state ρ_2, and then we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive unnatural value to be "2".

IV. THE INCOMPLETENESS IN A REAL EXPERIMENT

In a real experiment, there are no perfect detectors, but the good ones with some errors. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to even number $2N(\gg 1)$ and then we change such errors into trivial things. If we use the quantum predictions by even number $2N$ trials, then the inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

Again, we consider the noisy quantum states emerged from an imperfect source as follows:

$$\begin{aligned}
\rho_1 &= (1 - \epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}}, \\
\rho_2 &= (1 - \epsilon)|\downarrow\rangle\langle\downarrow| + \epsilon \times \rho_{\text{noise}}.
\end{aligned} \tag{13}$$

The value of $\epsilon(< 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\text{tr}[\rho_1\sigma_z] = +1 - \epsilon$ and $\text{tr}[\rho_2\sigma_z] = -1 + \epsilon$. Thus the measured results of trials are either $+1 - \epsilon$ or $-1 + \epsilon$.

The odd number results are $+1 - \epsilon$ by measuring the Pauli observable σ_z in the quantum state ρ_1 and the even number results are $-1 + \epsilon$ by measuring the same Pauli observable σ_z in the quantum state ρ_2. We suppose the number of trials of obtaining the result $-1+\epsilon$ is N that is equal to the number (N) of trials of obtaining the result $+1-\epsilon$. That is, the number of trials is even number $2N$.

We consider a following value $V_i (i = 1, 2, ..., N)$ which is the sum of two data in an optical experiment:

$$V_i = (+1 - \epsilon) + (-1 + \epsilon) = 0. \tag{14}$$

We introduce the following function $S(N)$ of even number $2N$ data in an optical experiment:

$$S(N) = (V_1 \times V_1) + (V_2 \times V_2) + ... + (V_N \times V_N). \tag{15}$$

If the number of trials is even number $2N$, then we have

$$\begin{aligned}
&S(N) = (V_1 \times V_1) + (V_2 \times V_2) + ... + (V_N \times V_N) \\
&= N\Big\{\Big((+1 - \epsilon) + (-1 + \epsilon)\Big) \times \Big((+1 - \epsilon) + (-1 + \epsilon)\Big)\Big\} \\
&= N\big(0 \times 0\big) = 0,
\end{aligned} \tag{16}$$

where we use $V_i = (+1 - \epsilon) + (-1 + \epsilon) = 0$. We derive a general and natural necessary condition of the function value $S(N)$. In this general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness, and we have

$$S(N) = 0. \tag{17}$$

This is the general and natural necessary condition when we consider both the proposition $[\sigma_z, \sigma_z] \neq 0$ and $[\sigma_z, \sigma_z] = 0$ and we assign simultaneously the different two values ("0" and "1") for the two propositions. We assign the value "0" for the proposition $[\sigma_z, \sigma_z] \neq 0$.

On the other hand, we can depict experimental data $r_1, r_2, r_3, ..., r_{2N}$ as follows: $r_1 = +1 - \epsilon$, $r_2 = -1 + \epsilon$, $r_3 = +1 - \epsilon$,..., $r_{2N} = -1 + \epsilon$. We can write V_i as follows:

$$V_i = r_{2i-1} + r_{2i}, \tag{18}$$

where

$$r_{2i-1} = +1 - \epsilon, \ r_{2i} = -1 + \epsilon. \tag{19}$$

Thus, we have

$$V_i = (+1-\epsilon) + (-1+\epsilon). \tag{20}$$

In the following, using commuting observable algebra, we evaluate another value to $V_i \times V_i$ and derive a specific and unnatural necessary condition for the function value $S(N)$ under the supposition that the two measured observables are commuting (that is, $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it) and we do not consider the existence of the following proposition $[\sigma_z, \sigma_z] \neq 0$. In this specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the sum rule is equivalent to the product rule [14]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have, after commuting observable algebra, (See (11)),

$$V_i \times V_i = 2(+1-\epsilon)^2. \tag{21}$$

Thus, we have

$$\begin{aligned} S(N) &= (V_1 \times V_1) + (V_2 \times V_2) + \ldots + (V_N \times V_N) \\ &= 2(+1-\epsilon)^2 + 2(+1-\epsilon)^2 + \ldots + 2(+1-\epsilon)^2 \\ &= 2N(+1-\epsilon)^2. \end{aligned} \tag{22}$$

This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. In this specific case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (17) and (22) when we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. We derive the inconsistency when we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it.

If we use the quantum predictions by even number $2N$ trials, then the inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We note our argumentations here agree with the discussions in Section III when $N = 1$. From the relation (17), we have the following general and natural value:

$$S(1) = 0. \tag{23}$$

From the relation (22), using commuting observable algebra, we have the following specific and unnatural value:

$$S(1) = V_1 \times V_1 = 2(+1-\epsilon)^2. \tag{24}$$

Thus, our discussions here are a natural expansion of Section III.

In summary, we have been in the inconsistency when measurement outcomes are even number $2N(\gg 1)$, the odd number results are $+1-\epsilon$ by measuring the Pauli observable σ_z in the quantum state ρ_1, the even number results are $-1+\epsilon$ by measuring the same Pauli observable σ_z in the quantum state ρ_2, and then we consider the existence of only the following proposition $[\sigma_z, \sigma_z] = 0$ and we assign the value "1" for it. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive unnatural value to be "2".

V. CONCLUSIONS

In conclusions, we have studied the mathematical foundations of quantum optics. We have considered the case that the uncertainty principle exists because we think both of commutativeness and non-commutativeness and then we derive natural value. On the other hand, we have considered the case that the uncertainty principle does not exist because we think only commutativeness and then we derive unnatural value. We have proposed an experimental accessible inconsistency within quantum optical phenomena in terms of imperfect sources and detectors. In more detail, we have encountered an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable in a real experimental situation. Such an error of the number of particles has become less and less important as we increase trials more and more by using the strong law of large numbers. As a result of our study, a perfect mathematical model for quantum optical phenomena does not have existed, at this stage.

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The authors are in an applicable thought to ethical approval.

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The authors state that there is no conflict of interest.

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