A Proof of Fermat’s Last Theorem
by Relating to Monic Polynomial Properties

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Abstract: Fermat’s Last Theorem (FLT) states that there is no natural number set \{a, b, c, n\} which satisfies \(a^n + b^n = c^n\) or \(a^n = c^n - b^n\) when \(n \geq 3\). In this thesis, we related LHS and RHS of \(a^n = c^n - b^n\) to the constant terms of two monic polynomials \(x^n - a^n\) and \(x^n - (c^n - b^n)\). By doing so, we could inspect whether these two polynomials can be identical when \(n \geq 3\), i.e., \(x^n - a^n = x^n - (c^n - b^n)\), which satisfies \(a^n = c^n - b^n\). By inspecting the properties of two polynomials such as factoring, root structures and graphs, we found that \(x^n - a^n\) and \(x^n - (c^n - b^n)\) can’t be identical when \(n \geq 3\), except when trivial cases.

1. Introduction

FLT was inferred in 1637 by Pierre de Fermat, and was proved by Andrew John Wiles [1] in 1995. But the proof is not easy even for mathematicians, requiring more simple proof.

Let \(a, b, c, n\) be natural numbers, otherwise specified. We related FLT to the following two monic polynomials.

\[
f(x) = x^n - a^n.
\]

\[
g(x) = x^n - (c^n - b^n).
\]

If \(f(x) = g(x)\) is possible for \(n \geq 3\), \(a^n = c^n - b^n\) is satisfied, and FLT is false. But the factoring, root structure and graph properties of \(f(x)\) and \(g(x)\) do not allow \(f(x) = g(x)\) when \(n \geq 3\). So, \(a^n = c^n - b^n\) can’t be satisfied for \(n \geq 3\).

2. Basic Lemmas

The number of roots of \(x^n - a^n\) is as follows, as in Figure 1 [2][3][4].

1. **Odd \(n \geq 3\):** One integer root and \(n - 1\) pairwise complex conjugate roots.

2. **Even \(n \geq 4\):** Two integer roots and \(n - 2\) pairwise complex conjugate roots.

![Figure 1. Number of roots examples of \(x^n - 1^n\).](image)

(a) Roots of \(x^5 - 1^n = 0\).  
(b) Roots of \(x^6 - 1^n = 0\).

**Lemma 2.1.** Below (2.1) is the irreducible factoring of (1.1) over the complex field [5].

\[
f(x) = x^n - a^n = \prod_{k=1}^{n} (x - ae^{\frac{2k\pi i}{n}}).
\]
Proof. The $n$ roots of (1.1) are $ae^{2k\pi i/n}, 1 \leq k \leq n$, so, (2.1) is the irreducible factoring of (1.1) over the complex field.

**Lemma 2.2.** Below (2.2) is the irreducible factoring of $h(c, b) = c^n - b^n$ over the complex field.

$$h(c, b) = c^n - b^n = \prod_{k=1}^{n}(c - be^{2k\pi i/n})$$  \hspace{1cm} (2.2)

Proof. The $n$ roots of $h(c, b)$ are $c = be^{2k\pi i/n}, 1 \leq k \leq n$, so, (2.2) is the irreducible factoring of $h(c, b)$ over the complex field.

**Lemma 2.3.** All $n$ factors of (2.2) can’t have same magnitude.

Proof. The $n$ factors of (2.2) are $c - be^{2k\pi i/n}, 1 \leq k \leq n$. Each factor can be considered as the difference vector between $(c, 0)$ and $b(cos \frac{2k\pi}{n}, sin \frac{2k\pi}{n})$, as in Figure 2.

**Figure 2.** Vector factor examples of (2.2).

(a) $n = 5$ example.

(b) $n = 6$ example.

Because $|c - be^{2k\pi i/n}|$ is same only with its complex conjugate $|c - be^{-2k\pi i/n}|$, the magnitude of all factors of (2.2) can’t be same for all $k$.

**Lemma 2.4.** A polynomial whose roots are all factors in (2.2) is (2.3) below.

$$p(x) = \prod_{k=1}^{n}(x - (c - be^{2k\pi i/n})).$$  \hspace{1cm} (2.3)

Proof. The $n$ factors of (2.2) are $c - be^{2k\pi i/n}, 1 \leq k \leq n$, and they are all involved in (2.3) as individual root. So, $p(x)$ is a polynomial whose roots comprise all factors in (2.2).

**Lemma 2.5.** A polynomial with different root magnitude can’t be of the form $x^n - a^n, n \geq 3$.

Proof. The $n$ roots of $x^n - a^n$ are all located on a circle of radius $a$ in the complex plane. But, if the magnitude of $n$ roots is not all same, all roots can’t be located on a same circle. So, a polynomial with different root magnitude can’t be of the form $x^n - a^n, n \geq 3$.

Lemma 2.5 implies that $f(x) = g(x)$ can’t be achieved for $n \geq 3$, so, $a^n = c^n - b^n$ can’t also be satisfied.
3. Graphical Interpretation of FLT and Proving Lemma

For graphical interpretation of FLT, example graphs of $f(x)$ and $p(x)$ are shown in Figure 3.

\[
f(x) = x^n - a^n. \quad (1.1)
\]

\[
p(x) = \prod_{k=1}^{n} \{x - (c - be\frac{2k\pi}{n})\}. \quad (2.3)
\]

**Figure 3.** Example graphs of $f(x)$ and $p(x)$.

![Graphs for $n = 1$.](image1)

(a) Graphs for $n = 1$.

![Graphs for $n = 2$.](image2)

(b) Graphs for $n = 2$.

![Graphs for odd $n \geq 3$.](image3)

(c) Graphs for odd $n \geq 3$.

![Graphs for even $n \geq 4$.](image4)

(d) Graphs for even $n \geq 4$.

We get $f(x)$ by vertically moving $y = x^n$ by $-a^n$. We get $p(x)$ by horizontally moving $y = x^n$ by $c$ and vertically moving by $-(-b)^n$.

\[
p(x) = \prod_{k=1}^{n} \{(x - c) - (-be\frac{2k\pi}{n})\} = \prod_{k=1}^{n} \{X - (-be\frac{2k\pi}{n})\} = X^n - (-b)^n, X = x - c. \quad (3.1)
\]

In graph view, FLT is equivalent to the moving of $p(x)$ to overlap $f(x)$, to find possible solutions that satisfy $a^n = c^n - b^n$. Moving $p(x)$ is equivalent to varying the integer values $(b, c), b \leq a < c$, i.e., moving $p(x)$ vertically or horizontally by integer steps. When any of $(b, c)$ makes two graphs overlap, a solution $a^n = c^n - b^n$ is found, and FLT is false. To make two graphs overlap, the following two steps are required.

1. Horizontal movement that makes $X = x - c$ in (3.1) to be $X = x$, i.e., $c = 0$.

2. Vertical movement that makes constant terms $a^n$ and $c^n - b^n$ equal.

In Figure 3 (a), when $n = 1$, $p(x)$ always overlaps $f(x)$ for $a = c - b$. In Figure 3 (b), when $n = 2$, $p(x)$ overlaps $f(x)$ for Pythagorean triples, $a^2 = c^2 - b^2 = (c - b)(c + b)$. When $n = 1, 2$, all roots of $f(x)$ and $p(x)$ affect the $(x, y)$-intercepts of the graphs, and there are infinitely many solutions.
But, when $n \geq 3$, the advent of complex roots, which do not appear in graphs, makes situations quite different from those of when $n = 1, 2$. Figure 3 (c) and (d) show that when $p(x)$ overlaps $f(x)$, $a = c - b$ or $a^2 = c^2 - b^2$ should be satisfied, which contradicts to $a^n = c^n - b^n, n \geq 3$. This is because the complex roots can’t affect the $(x, y)$-intercepts of the graphs. So, any integer step movements of $p(x)$ can’t satisfy $p(x) = f(x)$ when $n \geq 3$.

When $n \geq 3$, moving $p(x)$ to overlap $f(x)$ is equivalent to making all $n$ roots in $\prod_{k=1}^{n}(c - be^{-\frac{2k\pi}{n}})$ same as those in $\prod_{k=1}^{n}ae^{-\frac{2k\pi}{n}}$. Hence Lemma 3.1.

**Lemma 3.1.** When $n \geq 3$, to make every $n$ roots in $\prod_{k=1}^{n}(c - be^{-\frac{2k\pi}{n}})$ exactly match to those in $\prod_{k=1}^{n}ae^{-\frac{2k\pi}{n}}$, $c = 0, a = -b$ must be satisfied.

**Proof.** The complex number identity states that if $x + iy = u + iv$, then $x = u, y = v$ [6]. To satisfy $\prod_{k=1}^{n}ae^{-\frac{2k\pi}{n}} = \prod_{k=1}^{n}(c - be^{-\frac{2k\pi}{n}})$, keeping all $n$ roots in LHS and RHS identical, $ae^{-\frac{2k\pi}{n}} = c - be^{-\frac{2k\pi}{n}}$ must be satisfied.

$$a(cos \frac{2k\pi}{n} + isin \frac{2k\pi}{n}) = c - b(cos \frac{2k\pi}{n} + isin \frac{2k\pi}{n}).$$

$$asin \frac{2k\pi}{n} = -bsin \frac{2k\pi}{n}, \quad a = -b.$$  

$$acos \frac{2k\pi}{n} = c - bcos \frac{2k\pi}{n}, \quad c = 0.$$  

So, $c = 0, a = -b$. 

Lemma 3.1 comprises above mentioned step ① and step ②, where step ① makes $c = 0$ and step ② makes $a^n = c^n - b^n = -b^n$. That is to say, only trivial solutions can satisfy $a^n = c^n - b^n$ for $n \geq 3$.

4. Conclusion

In this thesis, we related LHS and RHS of $a^n = c^n - b^n$ to the constant terms of two monic polynomials $x^n - a^n$ and $x^n - (c^n - b^n)$. By doing so, the proof of FLT is simplified to the proof of whether the two polynomials can be identical when $n \geq 3$. The properties of the two polynomials such as factoring, root structures and graphs showed that $x^n - (c^n - b^n) = x^n - a^n$ can’t be achieved for $n \geq 3$, hence $a^n \neq c^n - b^n$ for $n \geq 3$. When $n = 1, 2$, there can be infinitely many $x^n - a^n = x^n - (c^n - b^n)$ solutions, but when $n \geq 3$, the advent of the complex roots latches further solutions, except for trivial ones. That is to say, as for the solutions of $a^n + b^n = c^n, \ a + b = c$ is the first and last solution for odd $n$, and $a^2 + b^2 = c^2$ is the first and last solution for even $n$. 

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