Mirror composite numbers. Their factorization and their relationship with Goldbag conjecture.

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Abstract:

In this paper we present the concept of mirror composite numbers. Mirror composite numbers are composite numbers of the form 2n-p for some n positive natural number and p prime. We shall show that the factorization of these numbers have interesting properties in order to face the Goldbach conjecture [1][2] by the divide et impera method.

Definitions:
From now on, m and n are positive integer numbers, p and q are prime numbers.
All prime numbers p \( \geq 5 \) are of the form 6m+1 or 6m-1. A prime of the form 6m+1 is a right prime; a prime of the form 6m-1 is a left prime.
A mirror composite number is a composite number of the form 2n-p for some n and some prime p \( \geq 5 \).
Given a mirror composite 2n-p, if p=6m+1, i.e., if p is a right prime, 2n-p is a right mirror composite (r.m.c.)
Given a mirror composite 2n-p, if p=6m-1, i.e., if p is a left prime, 2n-p is a left mirror composite (l.m.c.).

Lemma 1.
Fixed n, if 3 is a factor of some l.m.c (respectively r.m.c.), 3 is a factor of every l.m.c. (r.m.c.) and 3 is not a factor of any r.m.c. (l.m.c)
Proof:
The difference between two l.m.c. (r.m.c.) is 6n. If 3 | m, 3 | m±6n. On the other hand, if 3 | 2n-(6m-1), then 3 \( \not| \) 2n-(6m+1) and viceversa.

Lemma 2.
Fixed n, if q≠3 is a prime factor of two different l.m.c. (respectively r.m.c.), the difference between them is a multiple of 6q so the minimum gap between two consecutive occurrences of factor q is 6q for all l.m.c. (r.m.c).
Proof:
If q | 2n-(6x-1) and q | 2n-(6y-1) exists z such that zq=6(x-y), so z is multiple of 6, given that q is a prime and q ≠ 2,3.
If q | 2n-(6x+1) and q | 2n-(6y+1) exists z such that zq=6(x-y), so z is
multiple of 6, given that q is a prime and q ≠ 2,3.

**Goldbach conjecture** states that for all n and all p such that 3≤p≤n, some 2n-p is a prime, i.e., not every 2n-p is composite.

Let’s assume for the sake of contradiction that exists n such that every 2n-p is composite. Then, 3 consecutive odd numbers, 2n-3, 2n-5 and 2n-7 are composite, so one and only one of them must be multiple of 3.

**Case A: 3 | 2n-7:**

3 | 2n-7 ⇒ 3 | 2n-(6m+1) for all m (**Lemma 1**). Every right mirror composite is a multiple of 3 and no left mirror composite is a multiple of 3. So all elements of the sequence:

2n-3, 2n-5, 2n-11, 2n-17, 2n-23, 2n-29, 2n-41, ..., 2n-q

where q ≥ 5 is a left prime, must be factorized. There are k consecutive primes p_i (i=1,2,3, ..., k) from p_1=5 to p_k, where p_k is the largest prime p_k ≤ √2n-5, available for that factorization.

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a..., let be 2n-a; the number containing the first occurrence of prime factor p_i in that sequence. Notice that:

For each p_i, a; is unique.

3≤a;≤2p_i+1.

For some i, a; = 3; for some i, a;=5; for some i, a;=11 MOD p_i; for some i, a;=17 MOD p_i; for some i, a;=23 MOD p_i; and so on.

2n-q, i.e., 2n-(6m-1), is composite if and only if exists i such that 6m-1≡a; mod p_i (**Lemma 2**).

Now, let’s state conditions in order to find some 2n-q with q=6m-1 and q inside the interval √2n-5 ≤ q ≤ n that can not be factorized:

1) q is a prime, i.e., q is not multiple of any p_i, so 6m-1≡ 0 mod p_i for all i.

2) There is no p_i factor available for 2n-q, so 6m-1 ≠ a; mod p_i for all i.

<table>
<thead>
<tr>
<th>Prime condition for 6m-1</th>
<th>No factor available condition for 2n-(6m-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6m ≡ 1 mod 5</td>
<td>6m ≡ (a_1+1) mod 5</td>
</tr>
<tr>
<td>6m ≡ 1 mod 7</td>
<td>6m ≡ (a_2+1) mod 7</td>
</tr>
</tbody>
</table>
6m \equiv 1 \mod 11 \\
6m \equiv 1 \mod 13 \\
………….. \\
6m \equiv 1 \mod p_k \\
6m \equiv (a_3+1) \mod 11 \\
6m \equiv (a_4+1) \mod 13 \\
………….. \\
6m \equiv (a_k+1) \mod p_k \\

Hence for each p, there are at least $p_i-2$ remainders moduli $p_i$ that fullfill the conditions. That amounts up to a minimum of $(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $3.5.9.11\ldots(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

It’s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-5} < 6m < n$$

So let’s prove that at least one in $3.5.9.11\ldots(p_k-2)$ solutions from $5.7.11.13\ldots p_k$ systems lies inside the aforementioned interval.

Let be $M$ the highest number of consecutive occurrences of $6m$ that do not fullfill the conditions. Is not easy to figure out the value of $M$, given the unpredictable nature of prime number distribution. But we can prove that exists an upper bound $S$ for $M$ such that for sufficient large $n$:

$$S < \left\lfloor \frac{n-\sqrt{2n-5}}{6} \right\rfloor$$

(1)

Given $p_k$, an upper bound for the total number of occurrences of each one of the two remainders moduli $p$ are $2 \left\lceil \frac{p_k}{p} \right\rceil$. So

$$S = 2\left(\left\lceil \frac{p_k}{5} \right\rceil + \left\lceil \frac{p_k}{7} \right\rceil + \left\lceil \frac{p_k}{11} \right\rceil + \left\lceil \frac{p_k}{13} \right\rceil + \ldots + \left\lceil \frac{p_k}{p_{k-1}} \right\rceil + 1 \right)$$

is an upper bound for $M$:

<table>
<thead>
<tr>
<th>k</th>
<th>$p_k$</th>
<th>M</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

1 For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for $M$ is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in $p_k$ days. What is the maximum number, as a function of $p_k$, of consecutive days off?
In turn:

\[
\left\lfloor \frac{p_k}{5} \right\rfloor + \left\lfloor \frac{p_k}{7} \right\rfloor + \left\lfloor \frac{p_k}{11} \right\rfloor + \left\lfloor \frac{p_k}{13} \right\rfloor + \ldots + \left\lfloor \frac{p_k}{p_{k-1}} \right\rfloor + 1 < \]

\[
\frac{p_k}{2} + \frac{p_k}{3} + \frac{p_k}{5} + \frac{p_k}{7} + \frac{p_k}{11} + \ldots + \frac{p_k}{p_{k-1}} + 1 =
\]

\[
p_k \left\{ \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{p_{k-1}} + \frac{1}{p_k} \right\}
\]

The series between brackets is the well known partial summation of the reciprocal of the primes whose divergence was proved by Euler in 1737 together with the relationship:

\[
\sum_{p \leq x} \frac{1}{p} \approx \log\log(x)
\]

Taking \(x = p_k\) and given that an upper bound for all \(x > e^4\) in (2) is \(\log\log x + 6\) [3] allows us to state:

\[
S < 2p_k(\log\log p_k + 6)
\]

Now it’s immediate to conclude, since \(p_k \leq \sqrt{2n-5}\), that (1) holds for, let’s say, every \(2n \geq 10^6\).

For every \(2n < 10^6\) the verification of the conjecture have already been settled.

That completes the demonstration.

Hence, for all \(2n\) such that \(3 \mid 2n-7\), i.e., for all \(2n \equiv 1 \mod 3\), exists some \(2n-q\) that can not be factorized, so \(2n-q\) is prime and the conjecture holds for all \(2n \equiv 1 \mod 3\).

**Case B:** \(3 \mid 2n-5\):

\(3 \mid 2n-5 \Rightarrow 3 \mid 2n-(6m-1)\) for all \(m\) (Lemma 1). So every left mirror composite is a multiple of 3 and no right mirror composite is a multiple of 3...

Following the same thought process than before, with \(q\) a right prime
of the form $6m+1$, it’s straightforward to conclude that the conjecture holds for all $2n$ such that $3|2n-5$, i.e., for all $2n \equiv 2 \mod 3$.

**Case C: $3|2n-3$:**

Interesting matter of forward research.

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**References:**

1. Christian Goldbach, *Letter to L. Euler, June 7 (1742).*