A Simple Proof
That $e^{p/q}$ is Irrational

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Abstract
Using a simple application of the mean value theorem, we show that rational powers of $e$ are irrational.

Introduction
Hermite proved that $e$ is transcendental in 1873 [3]. His proof has been improved over the years by several mathematicians. A similar evolution has not taken place for proofs that show the irrationality of rational powers of $e$. In this note, we use relatively recent transcendence techniques [4, 6] to prove that the powers of $e$ are irrational.

This approach may have pedagogical advantages in that it allows for the understanding of recent transcendental techniques, for both $e$ and $\pi$, in the simpler context of an irrationality proof. It also gives a nice use of the mean value theorem that is suitable for first-year calculus students.

$e^p$ is irrational
Assume, to the contrary, that $e^p = a/b$ with $a$, $b$, and $p$ positive integers.
Since factorial growth exceeds polynomial, we can choose a positive integer $n$ large enough that
\[ be^p p^{2n+1} < n!. \] (1)
Choose a value of $n$ satisfying (1) and define $f(x) = x^n(p-x)^n$. Define $P(x)$ as the sum of $f(x)$ and its derivatives; that is,

$$F(x) = f(x) + f'(x) + \cdots + f^{(2n)}(x).$$

Next, let $G(x) = -e^{-x}F(x)$. Then $G'(x) = e^{-x}f(x)$. Using the mean value theorem on the interval $[0, p]$, we know there exists $\zeta \in (0, p)$ such that

$$\frac{G(p) - G(0)}{p} = G'(\zeta),$$

or

$$\frac{-e^pF(p) + F(0)}{p} = e^{-\zeta}f(\zeta).$$

(2)

Now, multiplying both sides of (2) by $pe^p$ gives

$$-F(p) + e^pF(0) = pe^{p-\zeta}f(\zeta),$$

and then substituting $e^p = a/b$ and multiplying by $b$ gives

$$-bF(p) + aF(0) = bpe^{p-\zeta}f(\zeta).$$

(3)

We claim that the left side of (3) is an integer multiple of $n!$. When we repeatedly differentiate $f(x)$, we find that every term of every derivative includes either a factor of $x$ or a factor of $n!$. Similarly, each term includes either a factor of $(p-x)$ or a factor of $n!$. It follows that both $F(0)$ and $F(p)$ are integer multiples of $n!$, and so the left side of (3) is also an integer multiple of $n!$. A Leibniz table, developed in [7], shows these properties succinctly.

Meanwhile, the right-hand side of (3) is strictly positive, and it is at most $bp^{2n+1}e^p$. This follows as the maximum values of $x^n$ and $(p-x)^n$ on $(0, p)$ are both $p^n$, so that $f(\zeta)$ is bounded above by $p^{2n}$. The additional $p$ factor in $pbe^{p-\zeta}f(\zeta)$ gives the $2n + 1$ exponent. Therefore, by (1), the right side of (3) is strictly less than $n!$.

We have, then, a contradiction. An integer multiple of $n!$ is positive, but less than $n!$.

**$e^{p/q}$ is irrational**

To show that rational powers of $e$ are irrational, suppose to the contrary that $e^{p/q} = a/b$, where $p, q, a, b$ are positive integers. Then

$$(e^{p/q})^q = e^p = (a/b)^q,$$

and, as $(a/b)^q$ is rational, this contradicts the irrationality of $e^p$. 

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Further reading

To see how the techniques used in this article can be applied, with some modifications, to show the irrationality of $\pi$, see [7]. Readers interested in a transcendence proof for $e$ should give Herstein’s proof a try [4]. After mastering the transcendence of $e$, we are ready to approach the big brother and big sister of all these irrationality and transcendence proofs: the transcendence of $\pi$, which shows that you can’t square the circle. Hobson gives the history of attempts to square the circle from antiquity up to the proof of its impossibility [5]. Niven’s 1939 transcendence of $\pi$ proof [8] adds some further historical perspectives while giving a simplification of Lindemann’s original 1882 proof. Original proofs of $e$ and $\pi$ can be found in [1].

References


