Real Numbers: a new (quantum) look
... with a hierarchical structure

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Abstract

• The motivation for re-designing $R$, as a Number System with “quantum structure”, is briefly provided, including hints to applications.

• Rational numbers $Q$ have much more structure beyond the ordered field structure which leads to Real Numbers as a metric completion, essential in Analysis and Classic Geometry (continuum / Limits / Calculus etc.):
  A) Farey sequence filtration;
  B) One point compactification, part of the integral projective line $P^1 \mathbb{Z}$:, i.e. the rational unit circle: $Q \to S^1_Q$ (including Pythagorean triples!).

• A topological completion of $Q$ provides a new construction of the real numbers, compatible with the above structure, adequate for applications in Number Theory and Quantum Physics.

  It is based on continued fractions representation of real numbers, and their “universal” binary representations using the modular group.

• Background on continued fractions, modular group.
Genesis of this project ...

Quantization of Classical Physics is difficult because of the “Continuum” (Real Numbers) … Idea: Why not re-designing Physics as a “Discrete Theory” …

2005: Quantum Computing, (Quantum) Digital World Theory” etc.; left to do: “Quantize The Qubit”!

2020: Solution: Finite groups for Standard Model etc. proton and neutron states (baryons) as modular curves with finite structure (Platonic, Archimedian solids etc.) [6] (... almost there!).

... but how to Dispense of Real Numbers!? see [1].

2023: (Stumbled upon) Real Numbers → Continued Fractions → Integral Mobius transformations! (Modular group, again!), so DISCRETE Lorentz Transformations: FINITE SPACE-TIME and STATES!!

Solution: Redesign REAL NUMBERS!
What is "wrong" with the Reals Numbers?

Top 10 reasons (for now) are:
1) They do not describe real quantities (Quantum Physics required quantization etc.);
2) Resulted from an “over-reaction” to extending Number Systems:

\[ N \rightarrow Z \rightarrow Q \rightarrow R \rightarrow C \rightarrow H \rightarrow O \ldots \]

the only analytic step in the sequence; in parallel algebraic numbers were developed, AG-periods etc. and R phased out gradually;
3) In Sciences we need 2D, to include periodicity (electric circuits etc.) so we need C; but do we really need \( C = R[i] \)!
4) Physics is conformal; Q are ratios, “conformal” (\( Q \rightarrow P^1Z \rightarrow P^1Q \), but then \( R \) is metric. We need “new reals” that are fractional transformations, to preserve the “field theory” tradition when extending algebraic numbers!
5) Nature is conformal and discrete: we always have conjugate variables (2D-conformal transformations. including rotations) and a quantum unit (compactification) [2].
6) Numbers are “shadows” of Math-Objects: almost all transcendental numbers don’t have such associated shadows (they are ghosts of Cauchy completion: no Cauchy sequences to justify we need them!).

7) We need to move on from “ratios and fields” (e.g. \( \mathbb{Q} \) etc.) to homogeneous structures (e.g. theory of ideas) and equations, as “true relations” (and look at their symmetries); i.e. a geometric picture: projective spaces / Hopf algebras (parallel and serial addition; renormalization via R-H Problem and Birkhoff decomposition etc.) [2].

8) \( \mathbb{R} \) separates Math into Real Analysis and friends etc., and Number Theory and “friends” (Alg. NT, AG). The theory of p-adics is a theory in \( \text{char} = p \) only at “tangent level” \( (\mathbb{F}_p) \); otherwise it is a mixture of AG and Analysis (“char 0”).

9) Complete \( \mathbb{Q} \) “to the left” and get p-adic numbers; “to the right” yields \( \mathbb{R} \); there should be a way to join them: by relating the prime at infinity and finite primes.

...3.14...

10) We don’t really use \( \mathbb{R} \) anyways! (excepting School, of course!).
The Plan

Another “clue” is that we encounter the modular group everywhere! which is the symmetry group of the rationals \( Q \) (\( SL_2(Z) \) preserves lattices, are “symplectic transformations” and conformal in 2D etc.) ... So, we should probably find a way to “extend” the field \( Q \) with its symmetries \( Aut(Q) \), to the “groupoid” of fractions and relations between them \( P^1Q \): Farey graph / map; this opens a Geometrization Program ...

So, since we can’t just get rid of real numbers, design a “bridge” between \( R \) and a “new” representation of the reals: CF and canonical Modular Group representation.

\[
SL_2(Z) = \langle S, T \rangle \xrightarrow{CF} R, \quad \text{and complete it: } R_{CF} = \overline{SL_2(Z)}.
\]

\[
N \rightarrow Z \rightarrow Q \rightarrow \overline{SL_2(Z)}_{AG} \rightarrow R_{CF}[i] \rightarrow H \rightarrow O...
\]

Where Alg.-Geom. really means “as needed by the Theory”\(^1\) ... and don’t forget the extra structure of a groupoid, to use for modular curves with tessellation, SM etc.

\(^1\)... and push \( i \) inside!
Modality and a few Benefits

The idea is to use CF representation to represent real numbers as ST-sequences of the modular group, including duality in Haar Wavelet Analysis.

It allows to investigate periods in the context of the modular group, in a geometric setting (projective space etc.). In fact think that it is a re-thinking of what $C$ is ...

The geometric setting involves ant-podal map, “bifield structure” (Hopf algebra with duality) etc. A relation with modular forms is expected, and from the algebraic side, with $L$ – functions and galois groups. Galois groups ($\pi_1$, algebraic fundamental groups, abstract GG etc.) need “upgraded” to Hopf algebras of symmetries with duality (Hopf objects in a category). Then a Pontryagin duality may even turn into a Laglands duality etc.
... and A little History of Real Analysis

- **Fourier Analysis** 1800s (used also by Babylonians!?)⁴ is the study of functions (e.g. real), in terms of periodic functions: *Signal analysis* etc. is based on an *additional structure of the Reals*: *Translations* (i.e. \(Z \rightarrow R\)) and *dilations modulo* \(Z\): \(D_n(x) = nx \mod Z\).

- The next step is **Wavelet Analysis**, where a hierarchy of details is introduced via *translations* \(T(x) = x - 1\) and *scaling* by a powers of 2, \(S(x) = 2^k x\): “zooming in or out” on the details of a function (signal); this is heavily used in transmission of pictures on the Web (see how a large image is loaded by your browser - on a slow connection!).

- ... but 2 is a choice (like 10 in decimal representations)! *“Modular Group Analysis”* of functions (new area) is “Universal”, *just adding inversion* \(S(z) = 1/z\). And, it is Quantum Physics and Number Theory “friendly” (allows to understand the atomic world etc.).

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²Wiki: “An early form of harmonic series dates back to ancient Babylonian mathematics, where they were used to compute ephemerides (tables of astronomical positions)!!
Main idea and implications

- **Fourier Analysis:** $T(x) = x + 1$ inv., $D_a(x) = ax$ affine group;
- **Wavelet Analysis** introduces “zooming in/out” grading by powers of 2.

**Limitations:** fixes the *prime at infinity*, separating Real Analysis from Number Theory. It is not NT friendly ($D_n(x) = nx \mod Z$ “screambles” irrationals).

- **Modular Group Analysis** “adds” inversion $S(x) = 1/x$; advantages:
  1) Maps $\infty \to p$ prime: unifies Real Analysis and Number Theory;
  2) Adds a *scale structure* to the Reals (Farey & CF filtration etc.);
  3) From (1) we expect to “bring Riemann Hypothesis Home” (to NT; not just a *bridge to Weil’s RH / Deligne Th.* - see [11]).
  4) Provides a *natural Algebraic-Geometric framework for Quantization* [6] and beyond (finite structures for *Standard Model of Elem. Part. Phys.*).
Goals and Designing Plans

• Introduce a new construction of the Reals, generalizing the idea of Haar wavelets, and compatible with the modular group action on fractions;

• Study the “meaningful real numbers”: \( Q \subset Alg.\,Numbers \subset Periods \).

• Present background material on continued fractions and modular group (2D-congruence Arithmetic);

• Benefits: bring Analysis “home”, to Algebraic Number Theory and Geometry.
The Rational Numbers

with

The Farey Filtration
Farey Fractions

- Fractions in the interval $[0, 1]$ can be grouped by the size of their denominators (in reduced form):
  \[ F_1 = \{0/1, 1/1\}, \]
  \[ F_2 = \{0/2, 1/2, 1/1\} = F_1 \cup \{1/2\}, \]
  \[ F_3 = \{0/3, 1/3, 1/2, 2/3, 1/1\}, \]
  \[ F_3 = F_2 \cup \{1/3, 2/3\} \text{ etc.} \]

**Definition**

The Farey sequences of fractions $F_n$ are defined inductively:

a) $F_1 = \{0/1, 1/1\}$;

b) $F_n$ is defined by adjoining to $F_{n-1}$ the “new” (irreducible) fractions with denominator $n$, which are of the form $k/n$ with $\gcd(k, n) = 1$ (irreducible fractions of denominator $n$).

Ex. $F_6 = F_5 \cup \{1/6, 5/6\}$ (see [2] for more details).

- Abstract Algebra interpretation (briefly): a fraction $3/5$ can be mapped to $3 \in \mathbb{Z}/5\mathbb{Z}$; then $F_n \approx F_{n-1} \cup (\mathbb{Z}/n, \cdot)$, i.e. a disjoint union of the units $U(\mathbb{Z}/n)$ of the rings $(\mathbb{Z}/n, +, \cdot)$ ...
Filtrations vs. Grading ...

This provides a filtration of the rationals:

\[ Q = \bigcup_{n \in \mathbb{N}} F_n, \quad F_1 \subset F_2 \cdots F_{n-1} \subset F_n \cdots \]

• A grading structure on a vector space/ ring / field etc. is a much richer, but also rigid structure. Ex.:
A) Vector spaces: \( V = V_1 \oplus V_2 \oplus V_2...; \)
B) Polynomials: \( R[x] = R \oplus R \oplus R...; \quad P(x) = c_0 + c_1x + c_2x^2... \)

• A grading structure on a ring (e.g. polynomials) is equivalent to a derivation rule, i.e. DERIVATIVE, via the Power Rule!!

\[(x^3)' = 3x^2,\]

comes from the grading, no limits or Calculus needed!!
Topology from Filtration

A topological structure (what is “near”, limits etc.) can be defined using open sets, topology for short, or sequences: Sequential Space / Sequential Topology [2].

Main idea: Convergent Sequences $\leftrightarrow$ Topology.

Hence, instead of using a metric to define Cauchy sequences of rational numbers, we define the class of sequences cofinal with the Farey filtration, as “convergent” by definition.

The real number $x \in \mathbb{R}$ can be represented as continued fractions $CF(x)$, which in turn, can be represented as a sequence $W$ of the standard generators $U(z) = 1/z$ and $T(z) = z + 1$ of the modular group $SL_2(\mathbb{Z})$.

1) In this way the filtration structure of $\mathbb{Q}$ can be transferred to the Reals, and a natural depth of approximation defined, instead of using a metric.

2) The relation with $p$-adic numbers will be studied elsewhere.

(Details / proofs, will appear in the joint article with Anurag Kurumbail)
The Modular Group (MG)

A group everybody should know ... Why?
• Congruence arithmetic is about \( \mathbb{Z} \) and \( \mathbb{Z}/n \): 1D ...
• 2D-Congruence arithmetic is a “complexification” of the above: \( SL_2(\mathbb{Z}) \) conformal transformations on the rational circle \( S^1_Q = Q/\mathbb{Z} \), including Pythagorean triples ... (relating nice elementary topics).

Definition

The modular group \( MG = PSL_2(\mathbb{Z}) \) is the group of 2D-matrices with integer coefficients \( SL_2(\mathbb{Z}) \), modulo \( \pm 1 \).

To a modular transformation, we associate a (complex) fractional transformation (integral Mobius transformation):

\[
S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ a, b, c, d \in \mathbb{Z} \quad \rightarrow \quad T(z) = \frac{az + b}{cz + d}, \ z \in \mathbb{C}.
\]
Generators of MG & Farey Fractions

Ex. *Unit translation* $T(z) = z + 1$ (addition of 1 as a generator of $(\mathbb{Z}, +)$), and *geometric inversion* $S(z) = -1/z$ are generators of the modular group. On fractions: $T(m/n) = (m + n)/n$, $S(m/n) = -n/m$.

There are many interesting properties of MG and its action on Farey fractions, but not enough time now ...

$$T(2/1) = \frac{2}{1} + 1 = 3/2$$

Each arc denotes an action of a MT.
Continued Fractions (CF)

Briefly, to get to our goal: $\bar{Q} = R$ ...

Real numbers have a \textit{continued fraction} representation:

a) CF of a rational function is finite:

$$\frac{7}{5} = 1 + \frac{1}{\text{CF}(5/2)} = 1 + \frac{1}{[2 + 1/2]} = [1; 2, 2];$$

b) CF of a quadratic number is periodic, e.g.

$$\sqrt{2} = [1; 2, 2, 2...] = [1; (2)];$$

c) CF of algebraic numbers?

d) ... of (Algebraic-Geometric) Periods? e.g.:

$$\int_0^1 \frac{4}{x^2 + 1} = \pi, \quad \zeta(4) = \pi^4/90, \text{ where } \zeta(k) = \sum_n 1/n^k.$$
Modular Group Representation of Reals

• A CF of a real number $r$ defines a unique sequence $W = (k_1, k_2, k_3 \cdots)$ of $PSL_2(Z)$ elements, such that $r = \ldots T^{k_3} S T^{k_2} S T^{k_1}(0)$; e.g.:

$$W = (3, 2, 2) : r = T^3 \circ S \circ T^2 \circ S \circ T^2(0).$$

Computing: $T^2(0) = 2$, $S(2) = -\frac{1}{2}$, $T^2(1/2) = 1/2 + 2$ etc.

• Conversely, each $ST$ sequence $W$ defines a Dedekind cut, hence a real number [3].

**Theorem**

The correspondence $W \rightarrow r(W)$ is a bijection and compatible with the usual topology of $R$. 
... and Euclid’s Algorithm

[Connections with elementary Math ...]

This is just encoding Euclid’s algorithm, when comparing two integers $m$ and $n$, to find $gcd(m, n)$ [3]:

$$m = q_1 \ n + r_1, \ n = q_2 \ r_1 + r_2 \quad \text{etc.} \quad \text{Ex.} \ \frac{7}{5} = 1 + \frac{2}{5}, \quad \frac{5}{2} = 2 + \frac{1}{2}.$$  

Denote $E(m, n) = (n, r) \iff m = qn + r$. Here the fraction $m/n$ is represented as it should, as a pair $(m, n)^3$. Equivalently:

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$W = TST^2ST^2S \iff W = (1, 2, 2) \iff \frac{7}{5} = [1; 2, 2] \ CF.$$  

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$^3$The monoid-to-group construction: $Q = Z \times Z^\times / \sim$.  

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Real Numbers: a new

November 1, 2023 19 / 36
In other words ...

Rational numbers, as equivalence classes of pairs (irreducible representatives), have an MG-representation, with two digits $S, T$, or even resembling the decimal rep.:

$$W = TSTTSTT = 1.2.2, \quad "\text{multiple "dot" - rep.}"$$

A dot signifies the inversion $S$; compare with, e.g. $3.14$, from integer to fractional.

This provides a resolution depth, similar to a $p$-adic valuation, except it is universal (geometric), base independent:

$$\nu(W) = \# \text{ of } S'\text{s}, \text{ i.e. } "\text{dots}".$$ 

- The well-known theory of CF ensures convergence: even products and odd products define a Dedekind cut, when the MG Word (MG-sequence) is infinite.
- The canonical family of MG-Sequences define a sequential topology implying Cauchy convergence and usual topology, EXCEPT it has more
Work to be done ... 

- Define in analogy with convergence of series:
  \[ s = \sum_{1}^{\infty} a_n : \quad s_n = \sum_{1}^{n} a_j, \quad s = \lim_{n \to \infty} s_n, \]

but for partial products of the MG-word \( W \), and check the Axioms of a Sequential Topology:

\[ q_n = \prod_{j=1..n} W_j, \quad \lambda(W) := \text{Lim } q_n \text{ (Dedekind cut)}. \]

- The extension and intrinsic interpretation in terms of complex integral \( \text{Mobius transformations} \ PSL_2(Z[i]) \) on the Riemann sphere \( S^2 = CP^1 \) is left for later developments ...

\textit{Rethinking} : \textit{Complex Numbers} \( C = R[i] \) via MG.
... and p-adic Numbers (Sketch / Skip for now ...)

The p-adic numbers $\mathbb{Q}_p$ are the “other completions of $\mathbb{Q}$, conform Ostrovski Theorem. They are filtered fields.

They also have CF, hence expressible in terms of the Modular Group. The completion of $\mathbb{Q}$ relative to the absolute value $|x - y|$ (usual Real Numbers) is also referred to as the norm corresponding to the prime at infinity.

One expects to have a unification by considering the Riemann sphere and inversion; similar to “point at infinity” for meromorphic functions ...
Indeed, the projective space $CP^1$ (or $RP^1$, the circle) can be view as defining two charts ($S^2$ North Pole $= \infty$ and $S^2$ South Pole $= 0$), and two isomorphic fields (about 0 ind $\infty$), isomorphic under inversion $S(z) = 1/z$.

This allows to add additional structure to the adeles ...
Why bother rethinking R? (in brief)

- The *measurement process*, e.g. in Quantum Physics, shows the inadequacy of choosing an arbitrary unit of measurement\(^4\)
- Physics quantities have a *Natural Unit*, e.g. Planck’s constant \(h\), electric charge \(e\) etc., as if a *greatest common divisor* ...
- *Physics Laws* have evolved into *Algebraic-Geometric Period Laws* [5]:

  Foundations : Cohomological Physics “ +” Number Theory.

What physicists measure in experiments are *AG-Periods*, disguised as “general real numbers”, via the *quantity / unit traditional approach*! (e.g. Feynman scattering amplitudes, charges, actions, angular momentum etc.). The role of periods in Physics is supported by the presence of *fundamental dimensional constants* (see *Buckingham’s Pi-Theorem* [4]), e.g. \(\alpha, R\#\) etc. These are naturally expressible via the Modular Group approach to “numbers”, as *encoding a comparison process*.

\(^4\)Incorporating Pythagorean’s philosophy: “Number Rules the Univers”, atomism, Zeno etc.
Farey Filtration and Projective Line

Farey sequence is a filtration of rationals in the interval $[0, 1]$. When a farey fraction $r = \left( \frac{p}{q} \right) \in \mathbb{Q}$ (irreducible) is plotted as a pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, it encodes the rays of the projective integral line $\mathbb{Q} \rightarrow P^1 \mathbb{Z}$.

- There are interesting connections with Pythagorean triples and rational circle, Pell’s equation.
- It relates with density of visible points and Riemann zeta function ...
Better: 2/3 trees (other filtrations)

- Stern-Brocot Tree is a filtration for all rationals:

```
0/1
1
2/1
3/2
4/3
```

Property: \[ \sum_{k} \frac{1}{p_k q_k} = 1, \text{ Level N fractions.} \]

[Relation with serial/parallel addition?]
Calkin-Wilf Tree

- Calkin-Wilf tree (see Wiki):
Modular Group, Platonic solids, Dessins d’Enfant ...

Using $SL_2(\mathbb{Z})$ and its subgroups as additional structure (modular curves), provides models in Elementary Particle Physics [6].

**Problem.** When considering a congruence subgroup $\Gamma \rightarrow SL_2(\mathbb{Z})$ and its geometry in $\mathbb{C}$, defines “special real numbers” $CF \ mod \ \Gamma$. Are these periods of the associated Belyi map? Is there a connection with modular forms?
Applications to Algebraic-Geometry and the SM

The use of Reals obtained via the modular group has ties to important Algebraic-Geometric objects with applications to Number Theory and Elementary Particle Physics.

- For example, the *Farey map* (the above triangulation $\mathcal{M}_3$ of $C_+$) has reductions $mod \ n$ (as pairs and group action) which include *Platonic Solids* for $n = 3, 4, 5$ (see [12]):

For the relation with quark flavors: $u, d, s, c, t, b$ see [6, 7].
• Numbers are “shadows” of Sets (Cantor’s cardinal numbers), Abelian Groups and other \textit{algebraic-geometric structures}, which are used as Physics models ...

Various classes of numbers are in fact grouped as belonging to Algebraic or Geometric Theories!
1) Rational numbers;
2) Algebraic numbers;
3) Periods etc.
“Arithmetic’s 5 Operations”

So, what the German mathematician Martin Eichler supposedly said [10]: “There are five fundamental operations in mathematics: Addition, subtraction, multiplication, division and modular forms.” (jokingly) is quite for “real” ...

Bridging the “gap”: Some fun topics in elementary Arithmetic and Number Theory:

• Congruence arithmetic and divisibility tests; e.g. 700s AD Arabic Casting out Nines Error Test;
• Euclid’s gcd algorithm and Continued Fractions representations of numbers;
• Rational numbers, Farey fractions and graph (mix of arithmetic, graphs, linear transformations).
Conclusions

A different approach to Real Numbers allows to extend Fourier Analysis and Wavelet Theory, with implications in Math and Physics. This allows to “bring char 0 to Number Theory”: a new theory of Adeles. Helps understand the practical Number Systems:

\[ N \rightarrow Z \rightarrow Q \rightarrow Algebraic \rightarrow Geometric \ (Periods) \ldots \]

[Note: \( e \) and \( \pi \) are really special \ldots (To be continued)]

The general use of the new structure, MG-sequences representing real numbers is:

\textit{To understand a Real Number} \quad \rightarrow \quad \textit{Study it’s CF in } SL_2(Z)!

i.e. don’t forget \( R \), just “translate into modern language”.
Further developments

Some suggestions are included:

- Study classes of “real numbers” with periodic ST-representations modulo a congruence subgroup $\Gamma \to SL_2(\mathbb{Z})$. e.g. at level $p$: $\Gamma = SL_2(F_p)$; relations with the modular curves? its Hodge-de Rham periods?
- The MG representation of the Reals provides a fractional representation, extending the one for rational numbers; consequences? How is the extended real line related to adeles, via this MG/ ST-representation?
- Modular forms of level $n$ are translation invariant $f(T(z)) = f(z)$ (Fourier periodic) and intertwines the antipodal inversion with a shift in the grading of its series (integration vs. $1/n! \, d^n/z^n$):

\[
f(S(z)) = z^n f(z), \quad f(W(z)) = z^n \nu(W(z)) f(z),
\]

where $\nu(W(z)) = \# \text{ of } S'\text{'s (inversions)},$ is a generalization of a valuation. Consequences?
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Wikipedia: Farey fractions; Modular group; Projective space; Sequencial topology etc.


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