A Truly Easy Proof: Pi is Irrational

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Abstract

Using the derivative of an integer polynomial composed with Euler’s formula we prove that π is irrational.

Proof

Proofs of the irrationality of π are numerous [1], but none are as easy and direct as the following.

Theorem 1. π is irrational.

Proof. A simple case generalizes. Suppose \( f_3(x) = x^3 \) and consider the sum of its derivatives:

\[
F_3(x) = x^3 + 3x^2 + 3!x + 3!.
\]

It follows that \( F_3(0) = 3! \). Now consider

\[
F_3(0)e^x = 3! \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \sum_{k=4}^{\infty} \frac{x^k}{k!} \right)
\]

\[
= F_3(x) + 3! \sum_{k=4}^{\infty} \frac{x^k}{k!}
\]

\[
= F_3(x) + 3!(e^x - s_3(x)),
\]

where \( s_3(x) \) is a partial sum of \( e^x \).
Adding $F(0)$ and imagining $x = \pi i$, we have

$$ (e^x + 1)F_3(0) = F_3(0) + F_3(x) + 3!(e^x - s_3(x)) \quad (1) $$

$$ 0 = \frac{F_3(0) + F_3(x)}{3!} + (e^x - s_3(x)). \quad (2) $$

There is no reason to believe that for a general term of any polynomial this pattern would change. Nor is there any reason that all surviving non-zero coefficients of $F_n(r)$, $r$ a root of $f_n(x)$ would not have factors of the multiplicity of the root, if the coefficients of $f_n(x)$ are integers. Thus assuming $\pi = p/q$, we can use $x^3(qx - pi)^3$, for example, and these conditions are met. So (2) gives, using Euler’s formula in (1), $0$ is an integer plus a fraction less than 1, a contradiction. \hfill \Box

**References**