

Guessing that the Riemann Hypothesis is unprovable using Non-Standard Analysis

T.Nakashima

Abstract

Riemann Hypothesis has been the unsolved conjecture for 164 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is $1/2$. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I guess the independence (unprovability) of the Riemann Hypothesis. In this, I deal with propositions equivalent to the Riemann Hypothesis regarding the Möbius function. First, I define something called "the distorted Möbius function". Next, I consider the axiomatic system in which the Riemann Hypothesis holds and the axiomatic system in which "the distorted Möbius function" exists and the Riemann Hypothesis does not hold. Finally, I guess that the Riemann Hypothesis is unprovable.

1

I define Möbius function $\mu(n)$ as

$$\mu(n) := \begin{cases} 1 & \text{product of even defferent primes} \\ -1 & \text{product of odd defferent primes} \\ 0 & \text{when divisible by the square of a prime} \end{cases}$$

The Riemann Hypothesis is that the real part of the non-trivial zero of the ζ function is $1/2$. (Ivić[2]p44)

Theorem 1. (Ivić[2]p48, Titchmarsh[5] p370, Theorem 14.25)

$$\text{the Riemann Hypothesis} \Leftrightarrow \sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

The following is known as a short proof.
 $M(x)$ is defined as follows.

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \int_{x=0.1}^{\infty} x^{-s} d(M(x))$$

$d(M(x))$ is the Stieltjes' integral of $M(x)$.

$$= [M(x)x^{-s}]_{0.1}^{\infty} + s \int_{x=0.1}^{\infty} M(x)x^{-s-1} dx = s \int_{x=0.1}^{\infty} M(x)x^{-s-1} dx$$

Here, I analytically continue $\frac{1}{\zeta(s)}$ until this value is finite. When $\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ holds, this integral is finite when $Re(s) > 1/2$. I obtain that there are no zeros of the zeta function when $Re(s) > 1/2$. Furthermore, by combining this with a discussion of functional equations, I obtain the Riemann Hypothesis.

Conversely, if I assume the Riemann Hypothesis,

$$s \int_{x=0.1}^{\infty} M(x)x^{-s-1} dx$$

This is not infinite at $Re(s) > 1/2 + \epsilon > 1/2$

$$|M(x)| < Km^{\frac{1}{2}+\epsilon}$$

I obtain that ϵ can be arbitrarily small.

Proposition 1.

$$\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

I guess this proposition's unprovability. This proposition is equivalent to the Riemann Hypothesis (by theorem 1).

From now on, we will consider two axiomatic systems to create two models: one where the Riemann hypothesis is true and one where it is false. First, I consider the axiomatic system that real and complex numbers satisfy plus the Riemann Hypothesis true. This satisfies the completeness. On

the other hand, I consider the axiomatic system that Non-Standard Analysis satisfies. The concepts of infinity and infinitesimal ([1]), whose existence is allowed in the Non-Standard Analysis, do not satisfy the completeness. Here, infinitesimal is positive number smaller than any positive number, and infinity is the reciprocal of infinitesimal, a number larger than any finite number. The axiomatic system of Non-Standard Analysis contains the axiom that is existence of the infinitesimal. Clearly, Cauchy sequences never converge in the infinitesimal. (A example of Infinitesimal is $\langle 1/1, 1/2, 1/3, \dots \rangle \leq 1/n = \langle 1/n, 1/n, 1/n, \dots \rangle$. A example of infinity is $\langle 1, 2, 3, \dots \rangle \geq n = \langle n, n, n, \dots \rangle$ (Hard and Loeb [1]))

From now on I argue in the Non-Standard Analysis. I define the sum of “the distorted Möbius function” as $\sum_{n < P-1} \mu(n) + f(P)$ that is in the summation of Möbius function (up to some large prime P) add (or subtract) $f(P)$ instead of $\mu(P)$. (The sign of $f(P)$ can be chosen freely. This is due to the multiplicative nature of the Möbius function.) Using the Non-Standard Analysis, let $P \rightarrow \infty$ (P is a prime number). I define “the distorted Möbius function” at $P * M$ as $f(P) * \mu(M)$. (When M is divisible by P , I set it to 0.) This value can be determined from P to infinity with no contradiction. Since only the value at P is needed from now on, I only consider that the sum of “the distorted Möbius function” up to P can be determined freely. The sum of “the distorted Möbius function” at $P \rightarrow \infty$ (P is a prime number) can be determined arbitrarily. (By the way, the sum of “the distorted Möbius function” can be taken arbitrarily for any m that satisfies $P < m < 2P$.) (There are two forms that will be used later. First, $|\sum_{n < P-1} \mu(n) + f(P)| = KP^{\frac{1}{2} + \epsilon_0}$ and $P \rightarrow \infty$, $\epsilon_0 > 0$ is the infinitesimal. (P is a prime number). Next, $|\sum_{n < P-1} \mu(n) + f(P)| = KP^{\frac{2}{3}}$ and $P \rightarrow \infty$ (P is a prime number).

The validity of the sum of “the distorted Möbius functions” is shown by the short proof of Theorem 1 above: when $|f(P)| = P^\sigma$, for $Re(s) > \sigma + \epsilon'$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \lim_{P \rightarrow \infty} \left(\frac{\mu(1)}{1^s} + \frac{\mu(2)}{2^s} + \dots + \frac{f(P)}{P^s} \right)$$

is true. In other words, if I take $f(P)$ at P , then $\zeta(s)$ has a zero point at $Re(s) > 1/2$. It is assumed that the zero point is $s = 1/2 + \epsilon_0 + iIm(s)$ or $s = \sigma + \infty i$, but this argument is not accurate. Here, I will only discuss the fact that it is possible to take “the distorted Möbius function” instead of the Möbius function.

conjecture 1. *The Riemann Hypothesis is unprovable. In other words, the Riemann Hypothesis is an "independent proposition".*

Consider the Riemann Hypothesis for the Möbius function as stated in Proposition 1.

First, consider the case where the negative proposition of the Riemann Hypothesis can be proven in normal axiomatic system. This is one possibility. In the following, I assume that there is no counterexample to Proposition 1 in a finite range.

Next, I show that the positive proposition of the Riemann Hypothesis is unprovable.

As Model A, consider the case where

$$\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

is true for all m . This case is the real and complex number's axiomatic system plus the Riemann Hypothesis is true.

As Model B, consider in the Non-Standard Analysis.

$$\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon_0})$$

holdss for some infinitesimal $\epsilon_0 > 0$ and $m = \infty$ case. and the case where

$$\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

is not true for $m = \infty$.

Model A is the model the Riemann Hypothesis is true. Model B is the model the Riemann Hypothesis is false.

There is no counterexample to Proposition 1 in a finite range. Model A holds in a finite range. Model A is "consistent" in ZFC (K.Kunen[3] Introduction§1). (Model A consists of equations that are not contradictory to each other. In such a case, it is said to be "consistent". (K.Kunen[3] Introduction§1))

Let take the sum of "distorted Möbius functions" for $m = P$ well. Then, Model B is true when $m = \infty$. Model B is "consistent" in the Non-Standard Analysis.

In such a case, the Riemann Hypothesis is independent of theory and cannot be proven. These cases implies the Riemann Hypothesis is not enough to prove in the real and complex number case. (Note that the completeness is not used in the argument.) (K.Kunen[3] Introduction§1)

It is possible that the negative proposition of the Riemann Hypothesis is provable, or the positive proposition of the Riemann Hypothesis is unprovable.

2 Equivalent Propositions

I will state that the three equivalent propositions of the Riemann Hypothesis are unprovable.

Proposition 2.

$$\pi(x) = li(x) + O(x^{\frac{1}{2}+\epsilon})$$

In the Non-standard Analysis, I will consider the situation at $P \rightarrow \infty$, assuming that $\pi(x)$ is summed $g(P) > 1$ for a certain prime number P . Or

$$\pi(x) = O(m^{\frac{1}{2}+\epsilon_0})$$

holds some infinitesimal $\epsilon_0 > 0$ and $m = \infty$.

As with the Möbius function, it is also clear that two opposing models exist and that they are consistent. Therefore, it is impossible to prove. Of course, it is equally possible to disprove it.

Proposition 3.

$$\sigma(n) < e^\gamma n \log \log n \text{ (for } n > 5040)$$

In the Non-Standard Analysis, For a sufficiently large prime factor P of $\sigma(n)$, replace all divisors AP^s related to P with $AP^s \times h(P)^s (h(P) > 1)$. Note that this proposition is an "inequality", so a model in which only one number does not satisfy the inequality is sufficient. This is because, unlike the previous two, $\sigma(n)$ should not be treated as "quantitative". Similarly, two consistent models can be created. Of course, this can also be disproved by finding a case in which the inequality does not hold within a finite range.

Proposition 4. *The only non-trivial zero point of the Riemann Zeta function is on $Re(s) = 1/2$.*

In this case, it is a "conjecture" half. I think in the Non-Standard Analysis. For $Re(s) > 1/2$, there may exist a zero point of the Riemann Zeta function that can be written in the form $s = Re(s) + \infty i$.

I can only see in some calculations.

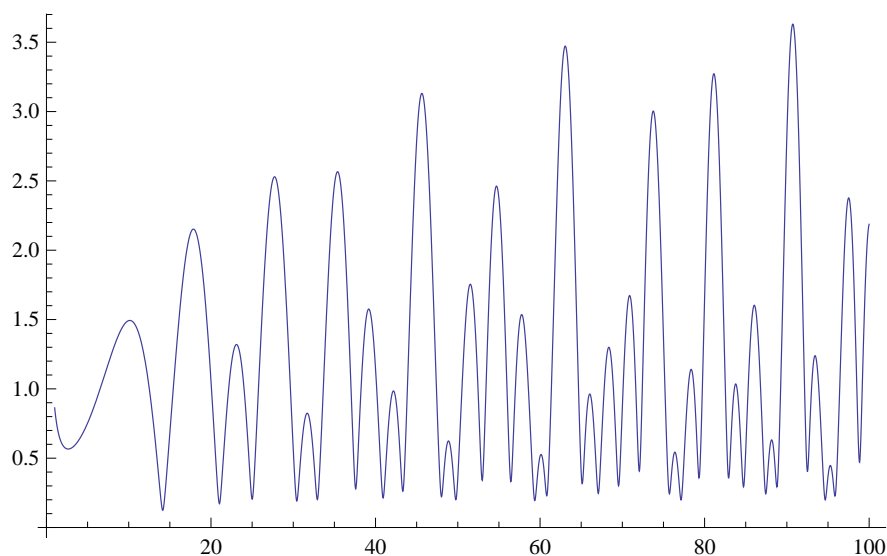


Figure 1: Plot [Abs [Zeta [2/3+s I]] ,(s, 1, 100)]

I have not examined all values, it may be gradually approach zero. Of course, there may be no example that satisfy this conjecture. If I replace the real value part $2/3$ with a number smaller than $1/2$, the result becomes even worse and it is hard to imagine that it approaches zero.

If $\zeta(s) = 0, Re(s) = 1/2$, then for infinitesimal ϵ_0 , we can set $\zeta(s + \epsilon_0) = 0, Re(s + \epsilon_0) \neq 1/2$. This thing means Riemann Zeta function is continue and holomorphic function at $s + \epsilon_0$ in the Non-Standard Analysis. This condition alone is sufficient to show that the Riemann hypothesis is unprovable. This impossibility of proving includes a scenario that I do not want to think about. It is impossible to prove, but it is possible to disprove it. This means that I can find a counterexample to the zero point from a small number, but the Riemann Hypothesis cannot be proven in this direction.

Note that the Riemann Hypothesis is true or false. However, in some axiomatic system, the Riemann Hypothesis do not hold. So, the positive proposition of the Riemann Hypothesis cannot prove, and the negative proposition of the Riemann Hypothesis is always provable.

Special thanks: I was very grateful to my friend H. Tokitu for translating in English. I would like to express my gratitude to him.

References

- [1] "An Introduction to Nonstandard Real Analysis", A. E. Hurd, P. A. Loeb. Pure and Applied Mathematics, A Series of Monographs and Textbooks, Columbia University, New York.
- [2] "The Riemann Hypothesis", Ivić. A, Dover Press
- [3] "Set Theory An Introduction To Independence Proofs", Kunen. K. Studies in Logic and the Foundations of Mathematics, Volume 102.
- [4] "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", Riemann Bernhard, Monatsberichte der Berliner Akademie. In Gesammelte Werke, Teubner, Leipzig (1892)
- [5] "The Theory of the Riemann Zeta function", Titchmarsh, E, C., Clarendon Press Oxford (1996)