The infinite series $1 + 2 + 3 + 4 + ...$ is strictly divergent

Amisha Oufaska

Abstract

In this paper, I prove that the infinite series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ...$ is strictly divergent or simply $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ... = \infty$ applying an argument by contradiction.

Notation and reminder

$\mathbb{N}^* := \{1,2,3,4, ... \}$ The natural numbers.

$\sum_{n=1}^{\infty} a_n = \infty$ means that the infinite series $\sum_{n=1}^{\infty} a_n$ is divergent.

$\mathbb{R}$: denoted the set of real numbers.

$\ln(x)$: denoted the natural logarithm of $x$.

Introduction

In analysis, a divergent series is an infinite series that is not convergent, meaning that the infinite sequence of the partial sums of the series does not have a finite limit. Before the 19th century, divergent series were widely used by Leonhard Euler and others, but often led to confusing and contradictory results. A major problem was Euler's idea that any divergent series should have a natural sum, without first defining what is meant by the sum of a divergent series. Augustin-Louis Cauchy eventually gave a rigorous definition of the sum of a (convergent) series, and for some time after this, divergent series were mostly excluded from mathematics. They reappeared in 1886 with Henri Poincaré's work on asymptotic series. In 1890, Ernesto Cesàro realized that one could give a rigorous definition of the sum of some divergent series, and defined Cesàro summation. Among the infinite series best known in mathematics, number theory and analysis we find the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ and the infinite series whose terms are the natural numbers $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ...$. 
1 + 2 + 3 + 4 + ⋯ = ∞

**Lemma** (Harmonic Series). \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty \).

*In other words, the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.*

**Proof.** \( \forall \ m \in \mathbb{N}^* \) we have \( \int_{1}^{m+1} \frac{dt}{t} \leq \sum_{n=1}^{m} \frac{1}{n} \Rightarrow \ln(m+1) \leq \sum_{n=1}^{m} \frac{1}{n} \)

and \( \lim_{m \to \infty} \ln(m+1) = \infty \Rightarrow \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \).

**Lemma.** \( \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n} \).

**Proof.** Indeed, \( \forall \ n \in \mathbb{N}^* \) we have \( \frac{1}{n} \leq n \) then \( \forall \ m \in \mathbb{N}^* \) we have

\( \sum_{n=1}^{m} \frac{1}{n} \leq \sum_{n=1}^{m} \frac{1}{n} \Rightarrow \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n} \leq \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n} \).

**Main Theorem.** \( \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots = \infty \).

*In other words, the series \( \sum_{n=1}^{\infty} n \) is strictly divergent.*

**Proof.** An argument by contradiction. Suppose that \( \sum_{n=1}^{\infty} n = \ell \in \mathbb{R} \) or the series \( \sum_{n=1}^{\infty} n \) is convergent. The Indian mathematician Srinivasa Ramanujan showed that \( \ell = -\frac{1}{12} \), he wrote about this in his letter to G. H. Hardy, British mathematician, dated 27 February 1913. On the other hand, \( \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} n \Rightarrow \infty \leq \ell \), and we get a contradiction.

**References**


E-mail address: ao.oufaska@gmail.com