Existing a prime in interval $n^2$ and $n^2 + \epsilon n$

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Abstract

Opperman’s conjecture states that there is a prime number between \( n^2 \) and \( n^2 + n \) for every positive integer \( n \), first we show that, all integer numbers between \( x^2 \) and \( x^2 + \epsilon x \) can be written as \( x^2 + i > 4p \) that \( 1 \leq i \leq \epsilon x \) and \( p = (x - m - 2)^2 + j \) in which \( j \) is a number in intervals \( 1 \leq j \leq \epsilon(x - m - 2) \), and then we prove generalization of Opperman’s conjecture i.e there is a prime number in interval \( n^2 \) and \( n^2 + \epsilon n \) such that \( 0 < \epsilon \leq 1 \).

Keywords: Bertrand-chebyshev theorem, Landu’s problems, Goldbach’s conjecture, twin prime, Legendre’s conjecture, Opperman’s conjecture

1. Introduction

Bertrand’s postulate state for every positive integer \( n \), there is always at least one prime \( p \), such that \( n < p < 2n \). This was first proved by Chebyshev in 1850 which is why postulate is also called the Bertrand-chebyshev theorem.

Legendre’s conjecture states that there is a prime between \( n^2 \) and \((n + 1)^2\) for every positive integer \( n \), which is one of the four Landu’s problems. The rest of these four basic problems are:

(i) Twin prime conjecture: there are infinitely many primes \( p \) such that \( p + 2 \) is a prime.

(ii) Goldbach’s conjecture: every even integer \( n > 2 \) can be written as the sum of two primes.

(iii) Are there infinitely many primes \( p \) such that \( p - 1 \) is a perfect square?

Problems (i), (ii), (iii) are open till date. Legendre’s conjecture is proved (in [8]).

Theorem: there is at least a prime between \( n^2 \) and \( n^2 + \epsilon n \), for every positive integer \( n \) such that \( 0 < \epsilon \leq 1 \) is constant arbitrary number.

To prove it by induction that if there is at least a prime between all \((x - 1)^2\) and \((x - 1)^2 + \epsilon(x - 1)\), then there is a prime between \( x^2 \) and \( x^2 + \epsilon x \).

To proceed to this proof, firstly we use the following Lemmas:

2. Lemmas

In this section, we present several lemmas which are used in the proof of our main theorem.

Lemma 2.1: for a large \( x \), all integer numbers between \( x^2 \) and \( x^2 + \epsilon x \) can be written as \( x^2 + i > 4p \) that \( 1 \leq i \leq \epsilon x \) and \( p = (x - m - 2)^2 + j \) in which \( j \) is a number in intervals \( 1 \leq j \leq \epsilon(x - m - 2) \), we assume that \( m = x/2 \) if \( x \) is even and \( m = (x + 1)/2 \) if \( x \) is odd and \( p \) is prime.

Proof: By induction there is a prime in intervals \( k^2 \) and \( k^2 + \epsilon k \) that \( k = a, (a + 1), ..., (x - 1) \) (for example if \( \epsilon = 1 \) so \( a = 2 \)), since \( m = x/2 \), if \( x \) is even or \( m = (x + 1)/2 \) if \( x \) is odd, so always \( p > (x - 4)^2/4 \), for a large \( x \), hence \( x^2 + i > 4p \), that \( 1 \leq i \leq \epsilon x \).

Lemma 2.2: If \( l \) to be the number of \( 3 \leq q < x \), (\( q \) is prime) are in equation
\[ x^2 + i = tq \text{ (is odd)} \] that \( 1 \leq i \leq cx \) so \( l < \frac{cx}{tq} \), for some \( 3 \leq q < x \)

Proof: if \( q \geq 3 \), we put \( i = j + 2ql \) (\( j \geq 1 \)), so \( j + 2ql \leq cx \), then \( l < \frac{cx}{tq} \) in this case \( l \) is the number of \( q \geq 3 \) that \( x^2 + i = tq \) is odd.

Lemma 2.3: If \( f \) to be the number of \( N > x \) are in \( x^2 + i = qN \) that these numbers are odd and \( 1 \leq i \leq cx \)

So:

For \( g = 3 \)

\[ f \leq \frac{cx}{2x3} \tag{1} \]

For \( g = 5 \)

\[ f \leq \frac{cx(1-1/3)}{2x5} \tag{2} \]

For \( g = 7 \)

\[ f \leq \frac{cx(1-1/3-1/5)}{2x7} \tag{3} \]

we continue this method to reach \( 1-1/3-1/5-\ldots-1/29 = almost 0 \) \( \tag{4} \)

Proof: If \( N > x \) and \( x^2 + i = qN \) to be odd, since \( 1 \leq i \leq cx \) so \( x^2/q \leq N \leq (x^2 + cx)/q \), in Which \( 3 \leq q < x \) are primes. Since the distance of between two odd numbers should be 2, so If \( q = 3 \), the number of such \( N \) odd number is:

\[ f \leq \frac{cx}{2x3} \]

but since \( N > x \), only one \( N > x \) could be in \( x^2 + i = qN \), so for \( q = 5 \),

\[ f \leq \frac{cx(1-1/3)}{2x5} \]

For \( q = 7 \),

\[ f \leq \frac{cx(1-1/3-1/5)}{2x7} \]

we continue this method to reach, \( 1-1/3-1/5-\ldots-1/29 = almost 0 \)

NOTE: we have only \( cx/2 \) composite odd numbers, since we say about \( N > x \) (this is new idea) not old idea i.e \( q < x \), for \( q = 3 \), we have \( cx/2 \) such \( N > x \), since we have only one such \( N > x \), exist, if we have two such primes i.e \( N1N2q > x^2 \) and this is contradiction, so for \( q = 5 \) \( cx/2 \) numbers changed to \( cx/2-cx/2/3 \), for \( q = 7 \) these numbers changed to \( (cx/2)-(cx/2)/3-(cx/2)/5 \) we continue this method to reach \( (cx/2)-(cx/2)/3-(cx/2)/5 \) \( -\ldots-(cx/2)/29 = almost 0 \) and also we have not same \( N \) for different \( q \), for example for \( q = 3 \), \( x^2/3 \leq N \leq (x^2 + cx)/3 \), for \( q = 5 \), \( x^2/5 \leq N \leq (x^2 + cx)/5 \), we can reach to

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contradiction notice that we consider numbers between $n^2$ and $n^2 + cn$.

3. The proof of main theorem

Theorem: There is at least a prime between $x^2$ and $x^2 + \varepsilon x$

Proof: Let we have at least a prime in intervals $k^2$ and $k^2 + \epsilon k$
that $k = a, (a + 1), ..., (x - 1)$ . By induction, we prove that, we have a prime between $x^2$ and $x^2 + \varepsilon x$. Assume that this is not true, so we can write $x^2 + i = 1 q, i.e all numbers in intervals 2 and $x^2 + \varepsilon x$ are not primes. Since $1 \leq i \leq \varepsilon x$ so according to ($G.H.Hardy, E.M.Wright, Oxford, 1964$) there is a prime factor like $q$ that for any composite number in $n^2$ and $n^2 + cn$ this interval $q \leq \sqrt{x^2 + \varepsilon x} \leq x + 1$

now we use the above results to reach to a contradiction, notice that we use odd statements so:

$$(x^2 + 1 \varepsilon 2)...(x^2 + ([\varepsilon x] - 1)or[\varepsilon x]) > (4p)^{\frac{\varepsilon x}{2}} (5)$$

According to lemmas 2.2 and 2.3, we have:

$$(x^2 + \varepsilon 2)...(x^2 + [\varepsilon x] - 1)or[\varepsilon x]) < 3^{\frac{x^2}{29}} \times ... \times 29^{\frac{x^2}{29}} \times \frac{x^2}{2} \times \frac{x^2}{5} \times \frac{(x^2)(1 - 1/2)}{29} \times \frac{x^2}{2} $$

We continue to reach $1 - 1/3 - 1/5 - 1/7 - 1/29 = almost 0 . Hence we have:

$$\frac{\varepsilon x - 2}{2} log(4p) < log(x^2 + 1 \varepsilon 2) + ... + log(x^2 + [\varepsilon x] - 1)or[\varepsilon x]) < (ex/2) \sum_{3 \leq q \leq w} \frac{log}{q} + (ex/2)(1/3 + (1 - 1/3)/5 + (1 - 1/3 - 1/5)/7 + ... + 0)logx^2 (7)$$

So by refer to $[3], \sum_{3 \leq q \leq w} \frac{log}{q} < logw + c$, that $c$ is positive constant number, so:

$$\frac{\varepsilon x - 2}{2} log(4p) < (ex/2)logw + (ex/2)c + 0.8(ex/2)logx^2 (8)$$

Then for a large $x$, $\frac{\varepsilon x - 2}{2} log(4p) < 1.7 \frac{\varepsilon x}{2} logx$, but since $\frac{\varepsilon x - 2}{2} > 0.94 \frac{\varepsilon x}{2}$ for a large $x$ so $p < x^{1/3}/4$ and this is a contradiction, because by lemma 2.1, $p > (x - 4)^2/4$.

References:


